11 Minkowski Polarity

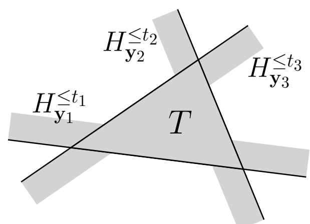
Recall: If $\mathbf{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, the *half-space* $H_{\mathbf{y}}^{\leq t}$ is given by

$$H_{\mathbf{y}}^{\leq t} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{y} \cdot \mathbf{x} \leq t\}.$$

Theorem 11.1 (Polytope Duality)**.** *Every polytope* $P = \mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ *can be expressed as an intersection of finitely many half-spaces: There exist* $\mathbf{y}_1, \ldots, \mathbf{y}_j$ *and* $t_1, \ldots, t_j$ *so that*

$$P = \bigcap_{i=1}^{j} H_{\mathbf{y}_i}^{\leq t_i}.$$

Example: The following triangle T in $\mathbb{R}^2$ is the intersection of the half-spaces $H_{\mathbf{y}_1}^{\leq t_1}, \ldots H_{\mathbf{y}_3}^{\leq t_3}$.

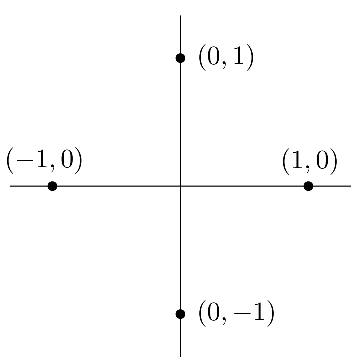


Minkowski discovered a beautiful way to realize this duality for polytopes and even more general sets. The following definition is his idea:

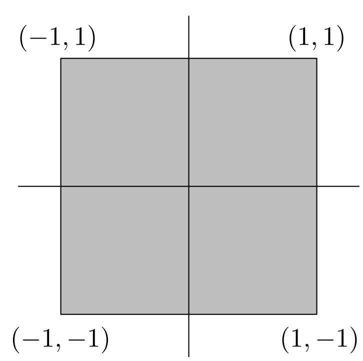
Definition. For any set $S \subseteq \mathbb{R}^n$, the *polar* of S is defined to be

$$S^\circ = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ holds for every } \mathbf{y} \in S\}.$$

Example 1:



$S = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$

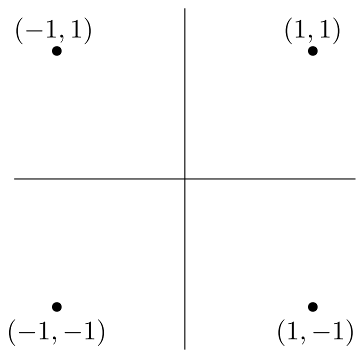


$S^\circ = \{\mathbf{x} \in \mathbf{R}^2 \mid \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for every } \mathbf{y} \in S\}$

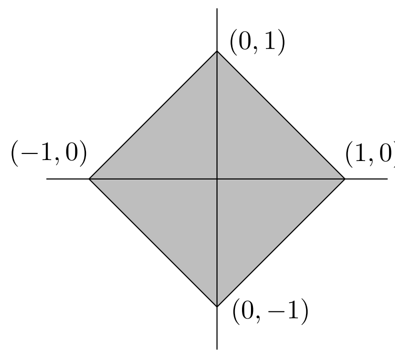
Note: For a finite set of points $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ we can represent the polar as an intersection of a finite list of half-spaces:

$$\begin{aligned}\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}^\circ &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y}_i \leq 1 \text{ for } 1 \leq i \leq k\} \\ &= H^{\leq 1}_{\mathbf{y}_1} \cap H^{\leq 1}_{\mathbf{y}_2} \ldots \cap H^{\leq 1}_{\mathbf{y}_k}.\end{aligned}$$

Example 2:



$T = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$



$T^\circ = \{\mathbf{x} \in \mathbf{R}^2 \mid \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for every } \mathbf{y} \in S\}$

The next lemma shows that the polar of a polytope is the same as the polar of its set of vertices.

Lemma 11.2. *If* $P = \text{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$*, then* $P^\circ = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}^\circ$.

Proof. It follows from the definition of the polar that every $\mathbf{x} \in P^\circ$ must satisfy $\mathbf{x} \cdot \mathbf{v}_k \leq 1$. Therefore $P^\circ \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}^\circ$. To complete the proof, we need to show $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}^\circ \subseteq P^\circ$. To do this, let $\mathbf{x} \in \mathbb{R}^n$ be an arbitrary point that satisfies $\mathbf{x} \cdot \mathbf{v}_i \leq 1$ for $1 \leq i \leq k$ and we will show that $\mathbf{x} \in P^\circ$. To show $\mathbf{x} \in P^\circ$ it suffices to check that $\mathbf{x} \cdot \mathbf{y} \leq 1$ holds for an arbitrary $\mathbf{y} \in P$. Since $\mathbf{y} \in P$ we have

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$$

for some coefficients $c_1, \ldots, c_k \geq 0$ with $c_1 + \ldots + c_k = 1$. Now we have

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot (c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k) \\ &= c_1(\mathbf{x} \cdot \mathbf{v}_1) + \ldots + c_k(\mathbf{x} \cdot \mathbf{v}_k) \\ &\leq c_1 + \ldots + c_k \\ &= 1\end{aligned}$$

and this completes the proof. □

Example 3: As a consequence of the above lemma and our first two examples we find that $P = \text{Conv}((0,1),(0,-1),(1,0),(-1,0))$ and $Q = \text{Conv}((1,1),(1,-1),(-1,1),(-1,-1))$ satisfy

$$P^\circ = \{(0,1),(0,-1),(1,0),(-1,0)\}^\circ = Q, \text{ and}$$
$$Q^\circ = \{(1,1),(1,-1),(-1,1),(-1,-1)\}^\circ = P$$

So, the polar operation takes P to Q and Q to P. The following theorem shows that this duality holds in much greater generality.

Theorem 11.3 (Minkowski)**.** *If $S \subseteq \mathbb{R}^n$ is closed and convex and contains $\mathbf{0}$ in its interior, then $(S^\circ)^\circ = S$.*

Proof. First we prove that $S \subseteq (S^\circ)^\circ$. To do so, let $\mathbf{y} \in S$. By the definition of the polar, every $\mathbf{x} \in S^\circ$ must satisfy $\mathbf{x} \cdot \mathbf{y} \leq 1$. However, it follows immediately from this that $\mathbf{y} \in (S^\circ)^\circ$. Next we prove that $(S^\circ)^\circ \subseteq S$. Suppose (for a contradiction) that this is not true and consider a point $\mathbf{w} \in (S^\circ)^\circ \setminus S$. Since S is closed and convex, there exists a hyperplane separating $\mathbf{w}$ and S. So, we may choose $\mathbf{z} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ satisfying:

1. $\mathbf{z} \cdot \mathbf{w} > t$ and

2. $\mathbf{z} \cdot \mathbf{y} \leq t$ for every $\mathbf{y} \in S$

Since $\mathbf{0}$ is in the interior of S there exists $\epsilon > 0$ so that $\epsilon \mathbf{z} \in S$. By the second property above we have $t \geq (\epsilon \mathbf{z}) \cdot \mathbf{z} = \epsilon ||\mathbf{z}||^2 > 0$. Now define the vector $\mathbf{z}' = \frac{1}{t}\mathbf{z}$ and note that we have

1. $\mathbf{z}' \cdot \mathbf{w} > 1$ and

2. $\mathbf{z}' \cdot \mathbf{y} \leq 1$ for every $\mathbf{y} \in S$

It follows from the second part above that $\mathbf{z}' \in S^\circ$. However, then we have a contradiction to $w \in (S^\circ)^\circ$ since $\mathbf{z}' \cdot \mathbf{w} > 1$. Therefore $(S^\circ)^\circ \subseteq S$, and this completes the proof. $\square$