10 Plane Graphs

Graph: A *graph* $G = (V, E)$ consists of a set of vertices V, a set of edges E, and an incidence relation so that each edge is incident with exactly two vertices, called its ends.

Connected: A graph is *connected* if it is possible to move from any vertex to any other vertex by traveling along vertices and edges.

Plane Graph: A *plane* graph is a graph drawn in the plane so that every vertex is a distinct point, and every edge with ends u, v is represented by a curve with endpoints corresponding to u, v so that none of these curves intersect one another except at the endpoints.

Faces: If $G = (V, E)$ is a plane graph, then G divides the plane into connected regions which are called *faces*. There is always one unbounded face called the *infinite* face.

Theorem 10.1 (Euler's Formula)**.** *If G is a connected plane graph with v vertices, e edges, and f faces, then $v - e + f = 2$.*

Proof: For any plane graph H we define $q(H)$ to be the number of vertices and faces of H minus the number of edges. Now, we shall proceed by reducing our graph G to a smaller graph without altering q.
First suppose that there is an edge e which separates the infinite face from another face. Then we may delete the edge e. This keeps the graph connected, and reduces e and f by one, so it does not affect q. By repeating this process, we reduce to a connected graph G' with just one face, the infinite face for which $q(G) = q(G')$.
Next suppose that there are at least two vertices. Then by walking along the graph without immediately reversing along an edge we cannot go in a cycle (this would imply > 1 face) so we must hit a dead end: a vertex x which is incident with just one edge, say s. Now delete x and s from the graph. This maintains connectedness and reduces v and e by one, so it does not alter q. By repeating this process we reduce to a graph G'' with just one vertex, just one face, and no edges. Now we have

$$q(G) = q(G') = q(G'') = 2$$

which completes the proof. $\square$

Platonic Solids

Definition. If v is a vertex of a graph, the *degree* of v is the number of edges incident with v. If F is a face of a plane graph, the *size* of F is the number of edges on the border of F.

Proposition 10.2. *Let G be a plane graph and assume that every vertex has degree d and every face has size k where $d, k \geq 3$. Then G is one of Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron.*

Proof: Assume that G has v vertices, e edges, and f faces. Now, we can count the number of pairs (x, s) so that x is a vertex and s is an edge incident with x in two ways. On one hand, each vertex has d incident edges, so this number is dv. On the other hand, each edge is incident with exactly two vertices so this number is also $2e$. Thus

$$2e = dv$$

Similarly, we can count the number of pairs (s, a) so that s is an edge and a is a face which is bordered by s. On one hand, each face is bordered by k edges so this number is kf. On the other hand, each edge borders two faces, so it is also $2e$. Thus

$$2e = kf$$

Substituting this into Euler's Formula yields

$$2 = v - e + f = \frac{2e}{d} - e + \frac{2e}{k}$$

so by elementary manipulations we have:

$$\frac{1}{d} + \frac{1}{k} = \frac{1}{e} + \frac{1}{2}$$

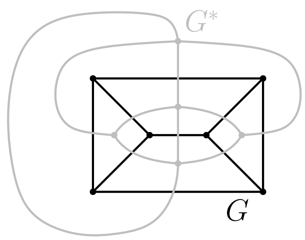
If $k, d \geq 4$ then $\frac{1}{d} + \frac{1}{k} \leq \frac{1}{2}$ so the above equation cannot be satisfied. Similarly, if one of d or k is equal to 3 and the other is ≥ 6, then $\frac{1}{d} + \frac{1}{k} \leq \frac{1}{2}$ so this equation cannot be satisfied. Thus, the only possible values for (d, k) are $(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$, and $(5, 3)$. Further, our equation implies that in these cases, G must (respectively) have 6, 12, 12, 30, and 30 edges. It then follows from an easy case analysis that Tetrahedron, Octahedron, Cube, Dodecahedron, and Icosahedron are the only possibilities. $\square$

Corollary 10.3. *The only Platonic solids are Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron.*

Duality

If G is a plane graph, one may form another plane graph G^* called the *dual* by the following process: Put one vertex of G^* in each face of G, and then whenever two faces F and F' of G have an edge e between them, add an edge to G^* between the two vertices corresponding to F and F' which crosses the edge e.

Example:



Observation 10.4. *Every plane graph G satisfies* $(G^*)^* = G$

So, vertices of the original graph correspond to faces of the dual, while faces of the original correspond to vertices of the dual. The edges of G and G^* are in one-to-one correspondence.

Note: Dualizing a Platonic solid yields another platonic solid. We have: