# 5 Affine Sets I

**Definition:** We say that a set $U \subseteq \mathbb{R}^n$ is *affine* if for every pair of distinct points $\mathbf{x}, \mathbf{y}$ in $U$ the line $\overleftrightarrow{\mathbf{xy}}$ is contained in $U$.

**Examples:** All of the following sets are affine

- Any single point
- A line
- A plane
- A hyperplane

**Observation 5.1.** *If $U$ is affine, then every translate of $U$ is affine.*

**Recall:** A set $U \subseteq \mathbb{R}^n$ is a subspace if it satisfies the following: Identity, closure scalars

**Lemma 5.2.** *If $U \subseteq \mathbb{R}^n$ and $\mathbf{0} \in U$, then $U$ is affine if and only if $U$ is a subspace.*

*Proof.* First we suppose that $U$ is a subspace and show that $U$ is affine. To prove that $U$ is affine, let $\mathbf{x}, \mathbf{y}$ be distinct points in $U$ and let $\mathbf{w}$ be an arbitrary point on the line $\overleftrightarrow{\mathbf{xy}}$ (we will show that $\mathbf{w} \in U$). Since $\overleftrightarrow{\mathbf{xy}} = \{\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \mid t \in \mathbb{R}\}$ we may choose $t \in \mathbb{R}$ so that $\mathbf{w} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$. By assumption $U$ is a subspace containing $\mathbf{x}$ and $\mathbf{y}$ so $(1-t)\mathbf{x} \in U$ and $t\mathbf{y} \in U$ and thus

$$\mathbf{w} = \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in U$$

Since $\mathbf{x}, \mathbf{y}$ and $\mathbf{w}$ were arbitrary, we deduce that $U$ is affine.
For the other direction we suppose that $U$ is affine and we will prove that it is a subspace. We have assumed that $\mathbf{0}$ is in $U$, so we already have the first property. To prove that $U$ is closed under scalar multiplication, we let $\mathbf{x} \in U$ and let $t \in \mathbb{R}$ and we will show that $t\mathbf{x} \in U$. This is automatically true when $\mathbf{x} = 0$ because in this case $t\mathbf{x} = t\mathbf{0} = \mathbf{0} \in U$. So we may assume $\mathbf{x} \neq \mathbf{0}$. Since $U$ is affine and $\mathbf{0}, \mathbf{x} \in U$ we have $\overleftrightarrow{\mathbf{0x}} \subseteq U$ and it follows that $t\mathbf{x} \in U$. It follows that $U$ is closed under scalar multiplication. To show that $U$ is closed under sums, we let $\mathbf{x}, \mathbf{y} \in U$ and we will show that $\mathbf{x} + \mathbf{y} \in U$. If $\mathbf{x} = \mathbf{y}$ then $\mathbf{x} + \mathbf{y} = 2\mathbf{x}$ is in $U$ since $U$ is closed under scalar multiplication. So, we may assume $\mathbf{x} \neq \mathbf{y}$. Since $U$ is affine it must contain the line

$$\overleftrightarrow{\mathbf{xy}} = \{\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \mid t \in \mathbb{R}\}$$

It follows that $U$ contains the point $\mathbf{w} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} = \mathbf{x} + \frac{1}{2}(\mathbf{y} - \mathbf{x})$. Since we have already proved $U$ is closed under scalar multiplication, $U$ must also contain the point $2\mathbf{w} = \mathbf{x} + \mathbf{y}$ as desired. □

**Theorem 5.3.** *For every nonempty affine set $U$ there is a unique subspace $V$ so that $U$ is a translate of $V$:*

$$U = \mathbf{u} + V$$

*Here the point $\mathbf{u}$ may be chosen to be any point in $U$.*

*Proof.* Let $U$ be a nonempty affine set and choose $\mathbf{u} \in U$. Now $V = -\mathbf{u} + U$ is an affine set containing $\mathbf{0}$, so $V$ is a subspace and we can write $U$ as $U = \mathbf{u} + V$ thus giving the desired equation.

For the second part, let $\mathbf{u}'$ be an arbitrary point in $U$. We wish to prove

$$\mathbf{u}' + V = \mathbf{u} + V$$

Since $U = \mathbf{u}' + V$ there exists $\mathbf{v} \in V$ so that $\mathbf{u}' = \mathbf{u} + \mathbf{v}$. Now, to prove that $\mathbf{u}' + V = \mathbf{u} + V$ we will show that each set is contained in the other.

(1) $\mathbf{u}' + V \subseteq \mathbf{u} + V$

To prove (1), we let $\mathbf{w}$ be an arbitrary point in $\mathbf{u}' + V$ and we will show $\mathbf{w} \in \mathbf{u} + V$. To do this, express $\mathbf{w}$ as $\mathbf{w} = \mathbf{u}' + \mathbf{v}'$ where $\mathbf{v}' \in V$. Now using the fact that $V$ is a subspace we have

$$\mathbf{w} = \mathbf{u}' + \mathbf{v}' = \mathbf{u} + \mathbf{v} + \mathbf{v}' \in \mathbf{u} + V$$

(2) $\mathbf{u} + V \subseteq \mathbf{u}' + V$

To prove (2), we let $\mathbf{z}$ be an arbitrary point in $\mathbf{u}' + V$ and we will show $\mathbf{z} \in \mathbf{u} + V$. To do this, express $\mathbf{z}$ as $\mathbf{z} = \mathbf{u} + \mathbf{v}''$ where $\mathbf{v}'' \in V$. Now using the fact that $V$ is a subspace we have

$$\mathbf{z} = \mathbf{u} + \mathbf{v}'' = \mathbf{u}' - \mathbf{v} + \mathbf{v}'' \in \mathbf{u}' + V$$

□

**Definition:** If $U$ is a nonempty affine set, then we define the *dimension* of $U$ to be the dimension of the subspace $V$ from the above theorem.

**Examples:**

- Affine sets of dimension 0 are sets consisting of a single point.
- Affine sets of dimension 1 are lines
- Affine sets of dimension 2 are planes
- Affine sets of dimension $n - 1$ are hyperplanes