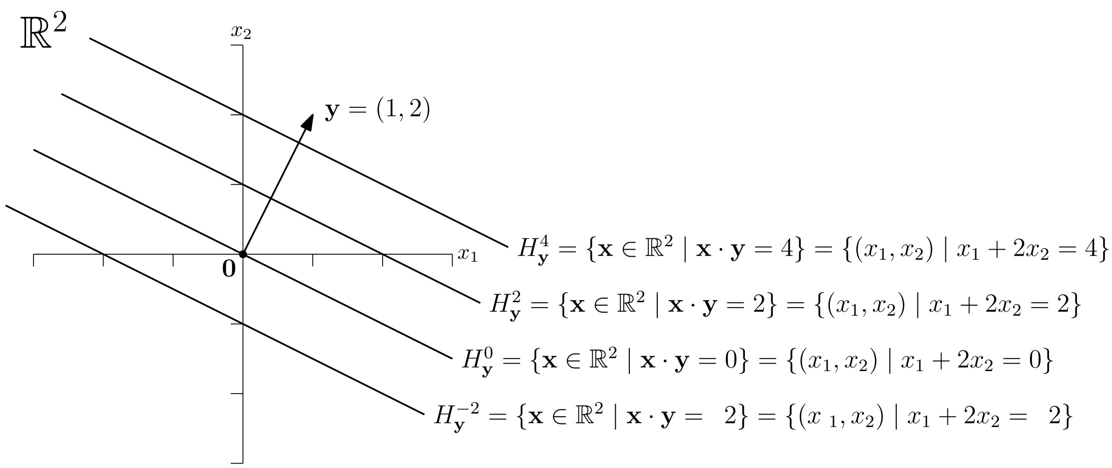
4 Hyperplanes

Definition: If $\mathbf{y} \in \mathbb{R}^n$ is nonzero and $\mathbf{t} \in \mathbb{R}$ we define the set

$$H_{\mathbf{y}}^t = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} = t\}$$

and we call any set of this form a *hyperplane*.

Example.



Note: Consider the point $(0, 1)$ on the hyperplane $H_{\mathbf{y}}^2$ above. Suppose we want to move from $(0, 1)$ by adding a vector $\mathbf{v} = (v_1, v_2)$ to it. How can we arrange that this new point $(0, 1) + (v_1, v_2)$ lies on the same hyperplane $H_{\mathbf{y}}^2$? A quick check reveals that we need for the vector $\mathbf{v}$ we add to be orthogonal to $\mathbf{y}$. Next we show that this happens in general.

Lemma 4.1. *Let* $\mathbf{w}$ *be a point on the hyperplane* $H_{\mathbf{y}}^t$ *and let* $\mathbf{v} \in \mathbb{R}^n$*. The point* $\mathbf{w} + \mathbf{v}$ *is on the hyperplane* $H_{\mathbf{y}}^t$ *if and only if* $\mathbf{y} \cdot \mathbf{v} = 0$

Proof. This is a rather direct consequence of the computation

$$\big(\mathbf{w} + \mathbf{v}\big) \cdot \mathbf{y} = \mathbf{w} \cdot \mathbf{y} + \mathbf{v} \cdot \mathbf{y} = t + \mathbf{v} \cdot \mathbf{y}.$$

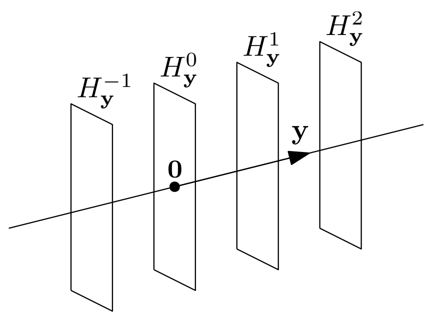
If $\mathbf{v}$ is orthogonal to $\mathbf{y}$ then $(\mathbf{w} + \mathbf{v}) \cdot \mathbf{y} = t$ so $\mathbf{w} + \mathbf{v}$ is on $H_{\mathbf{y}}^t$. If $\mathbf{v}$ is not orthogonal to $\mathbf{y}$, then $(\mathbf{w} + \mathbf{v}) \cdot \mathbf{y} \neq t$ so $\mathbf{w} + \mathbf{v}$ is not on $H_{\mathbf{y}}^t$. $\square$

Figure in $\mathbb{R}^3$ containing three points.

Show that the hyperplane is also of the form $\mathbf{x_1} + \text{Span}()$

Note: The line $\text{Span}(\mathbf{y})$ is precisely the set $\{s\mathbf{y} \mid s \in \mathbb{R}\}$. The dot product of $s\mathbf{y}$ and $\mathbf{y}$ equals $(s\mathbf{y}) \cdot \mathbf{y} = s\mathbf{y} \cdot \mathbf{y} = s||\mathbf{y}||^2$. So if we imagine moving along the line $\text{Span}(\mathbf{y})$ by starting at $s\mathbf{y}$ and increasing s we will increase the dot product of our point with $\mathbf{y}$. It follows from this and the above lemma that hyperplanes always appear as in the figure below.

Example.



Lemma 4.2. *The translation of a hyperplane $H_{\mathbf{y}}^t$ by a vector $\mathbf{v}$ is a hyperplane given by*

$$\mathbf{v} + H_{\mathbf{y}}^t = H_{\mathbf{y}}^{t+\mathbf{v}\cdot\mathbf{y}}$$

Proof. This lemma follows immediately from the following chain of equivalent statements.

$$\begin{aligned} \mathbf{x} \in \mathbf{v} + H_{\mathbf{y}}^t &\Leftrightarrow \mathbf{x} - \mathbf{v} \in H_{\mathbf{y}}^t \\ &\Leftrightarrow \big(\mathbf{x} - \mathbf{v}\big) \cdot \mathbf{y} = t \\ &\Leftrightarrow \mathbf{x} \cdot \mathbf{y} - \mathbf{v} \cdot \mathbf{y} = t \\ &\Leftrightarrow \mathbf{x} \cdot \mathbf{y} = t + \mathbf{v} \cdot \mathbf{y} \\ &\Leftrightarrow \mathbf{x} \in H_{\mathbf{y}}^{t+\mathbf{v}\cdot\mathbf{y}} \end{aligned}$$

$\square$