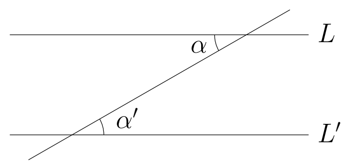
1 Basic Geometry and Trigonometry

Euclidean Plane

We begin with an obvious proposition which we will not prove formally. To see why it holds true, note that the figure has a rotational symmetry.

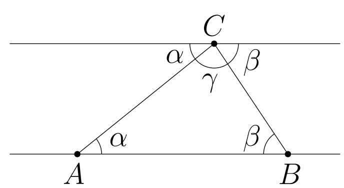
Proposition 1.1. *In the figure below, assuming the lines L and L' are parallel, the angles α and α' are equal.*



Definition: If $A, B \in \mathbb{R}^2$ are distinct, we let $\overleftrightarrow{AB}$ to denote the line containing them.

Theorem 1.2. *The angles from any triangle in the plane sum to π*

Proof. Consider a triangle with vertices A, B, C and construct a line through the point C which is parallel to the line $\overleftrightarrow{AB}$. The result follows by applying Proposition 1.1 to this figure.

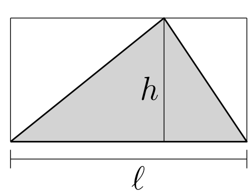


□

Recall: The area of a rectangle with length ℓ and height h is equal to ℓh.

Theorem 1.3. *The area of a triangle with length ℓ and height h is equal to $\frac{1}{2}\ell h$.*

Proof. Consider a triangle with length ℓ height h as shown in the figure below. The area of this triangle is half that of the outer rectangle and this gives the result.



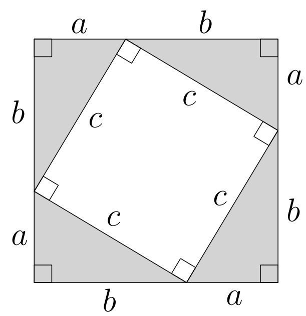
A similar argument applies when the line segment corresponding to the height falls outside the triangle. □

Recall: A *right* triangle is a triangle with one angle equal to $\frac{\pi}{2}$; the side opposite this angle is called the *hypotenuse*.

Theorem 1.4 (Pythagoras)**.** *If a right triangle has side lengths a, b, c where c is the length of the hypotenuse, then*

$$a^2 + b^2 = c^2.$$

Proof. Consider the following figure.

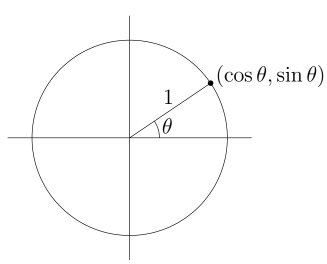


The area of the large outer square equals the sum of the areas of the four (identical) right triangles plus the area of the inner square. Therefore

$$(a+b)^2 = 4(\tfrac{1}{2}ab) + c^2 = 2ab + c^2$$

and the result follows immediately. □

Trigonometry



Definition. We define the functions sin and cos by the rule that the point on the unit circle at an angle of θ has cartesian coordinates $(\cos\theta, \sin\theta)$.

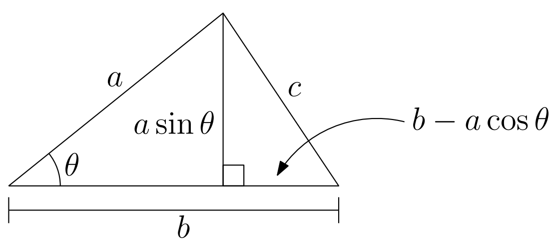
Notes:

- sin and cos are periodic with period 2π.
- For every $x \in \mathbb{R}$ we have $\sin^2 x + \cos^2 x = 1$.
- The point distance r from the origin with angle θ has coordinates $(r\cos\theta, r\sin\theta)$.

Theorem 1.5 (Law of Cosines)**.** *If a triangle has sides of length a and b with an angle of θ in between, then the length of the remaining side c satisfies*

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Proof. Align the triangle so that the vertex between the sides of length a and b is at the origin and the side of length b runs along the positive x-axis. Assume that the line segment corresponding to the height falls within the triangle as shown in the figure below.



Now the smaller triangle on the right in the above figure is a right triangle with hypotenuse of length c and the other two sides have length $a\sin\theta$ and $b - a\cos\theta$. Applying the Pythagorean Theorem to this triangle gives us

$$\begin{aligned} c^2 &= (a\sin\theta)^2 + (b - a\cos\theta)^2 \\ &= a^2\sin^2\theta + b^2 + a^2\cos^2\theta - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

□