the exterior vertices. These paths exist because a planar triangulation is 3-connected, but the paths obtained by this algorithmic process satisfy several key structural properties. Schynder's argument can be extended easily to show that given a planar map in which every face, except possibly the exterior face, is a triangle, the poset of vertices, edges, and faces ordered by inclusion has dimension at most four. However, his techniques do not apply to arbitrary planar maps.

Brightwell and Trotter then succeeded in showing that the vertex-edge-face poset of any planar map has dimension at most four even if loops and multiple edges are allowed. The problem of determining the dimension of the vertex-edge-face poset is first reduced to the consideration of planar maps which are ordinary graphs and are 2-connected. We then develop some combinatorial properties of a family of paths constructed in a planar graph satisfying a slightly weaker property than being 3-connected. The 3-connected planar graphs are of particular interest because these are just the planar graphs associated with convex polytopes in 3-dimensional Euclidean space. For such graphs, Brightwell and Trotter showed that the poset consisting of the vertices and faces ordered by inclusion is 4-irreducible. For $t \geq 4$, there are convex polytopes in t-dimensional Euclidean space for which there is no bound on the dimension of the vertex-edge poset.

2. Schnyder's Dimension Theoretic Test for Planarity

Let $\mathbf{G} = (V, E)$ be an ordinary graph. We associate with $\mathbf{G}$ a poset $\mathbf{P} = (X, P)$ called the *vertex-edge* poset of $\mathbf{G}$ (also, the *incidence* poset of $\mathbf{G}$) by $X = V \cup E$ and $x < y$ in P if and only if $x \in V$, $y \in E$, and x is an end point of y. I consider the next theorem to be one of the most significant results in dimension theory since the concept was introduced 50 years ago.

(2.1) Theorem (Schnyder [SCHN]). *Let* $\mathbf{G} = (V, E)$ *be a graph and let* $\mathbf{P} = (V \cup E, P)$ *be the vertex-edge poset associated with* $\mathbf{G}$. *Then* $\mathbf{G}$ *is planar if and only if* $\dim(\mathbf{P}) \leq 3$.

Proof. Suppose first that $\dim(\mathbf{P}) \leq 3$. We show that $\mathbf{G}$ is planar. (This is the relatively easy part of Schnyder's theorem, and the argument presented here is actually due to Babai and Duffus [BA-DU].) We use the well-known fact that a graph $\mathbf{G} = (V, E)$ is planar if it can be drawn in the plane so that there are no edge crossings involving edges e_1, e_2 which do not share an end point. Edge crossing involving edges with a common end point present no problem as such crossings can be eliminated.

Choose an embedding of **P** in $\mathbf{R}^3$ which associates with each $y \in V \cup E$ a vector $\mathbf{y} = (y_1, y_2, y_3) \in \mathbf{R}^3$ so that $u \leq v$ in **P** if and only if $u_i \leq v_i$ in **R** for $i = 1, 2, 3$. For each $y \in V \cup E$, let $\pi(y)$ be the orthogonal projection of **y** on the plane $x_1 + x_2 + x_3 = 0$ in $\mathbf{R}^3$. Without loss of generality, all points in $V \cup E$ project to distinct points on the plane $x_1 + x_2 + x_3 = 0$, and these points are in general position.

For each $u \in V$ and each $e \in E$ containing u as an end point, join $\pi(u)$ and $\pi(e)$ with a straight line segment. If **G** is nonplanar, there exist distinct vertices $u, v \in V$ and distinct edges $e, f \in E$ so that u is an end point of e but not of f, v is an end point of f but not of e, and the line segment $\pi(u)\pi(e)$ crosses the line segment $\pi(v)\pi(f)$ at a point **p** interior to both. Let **z** be the point on the line segment **ue** in $\mathbf{R}^3$ so that $\pi(\mathbf{z}) = \mathbf{p}$. Also let **w** be the point on the line segment **vf** in $\mathbf{R}^3$ so that $\pi(\mathbf{w}) = \mathbf{p}$. Then either $\mathbf{z} \leq \mathbf{w}$ in $\mathbf{R}^3$ or $\mathbf{w} \leq \mathbf{z}$ in $\mathbf{R}^3$. However, $\mathbf{z} \leq \mathbf{w}$ implies $\mathbf{u} \leq \mathbf{z} \leq \mathbf{w} \leq \mathbf{f}$, which is false since u is not an end point of f. Similarly $\mathbf{w} \leq \mathbf{z}$ implies $\mathbf{v} \leq \mathbf{e}$ which is also false. The contradiction shows that **G** is planar.

Now suppose that **G** is planar. We show that the vertex-edge poset **P** has dimension at most 3. Without loss of generality, we assume that **G** is maximal planar since adding edges to **G** can only increase the dimension of the associated vertex-edge poset. Choose a planar drawing of **G** using straight line segments for the edges. This diagram is a triangulation T of the plane. Each interior region is a triangle, and T has three exterior vertices which we label in clockwise order v_1, v_2, and v_3.

Now consider a function f which assigns to each angle of each interior triangle of T a color selected from $\{1, 2, 3\}$. The function f is called a *normal* coloring of T if the following properties are satisfied:

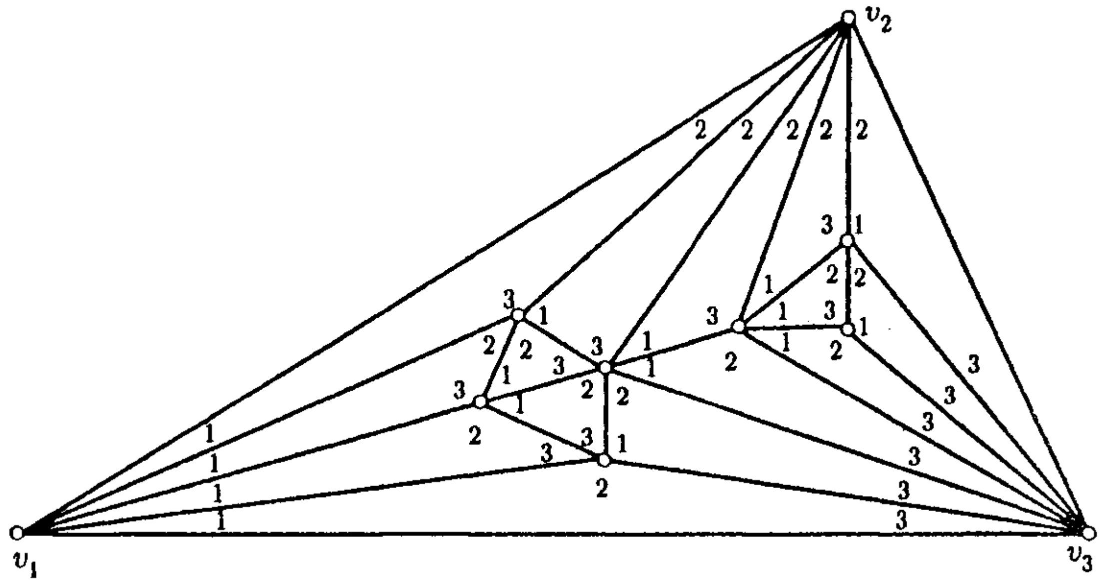
(1) All angles incident with exterior vertex v_i are mapped by f to color i, for $i = 1, 2, 3$;

(2) At each interior vertex u of T, there is an angle mapped by f to color i, for $i = 1, 2, 3$;

(3) At each interior vertex u of T, all angles mapped by f to color i are consecutive, for $i = 1, 2, 3$;

(4) At each interior vertex u of T, the block of angles mapped by f to color 2 appears immediately after the block of angles mapped by f to color 1 in clockwise order; and

(5) For each elementary triangle of T, f assigns the three angles to colors 1, 2, and 3 in clockwise order. We illustrate this definition in the following figure with a normal coloring of a triangulation.



The following claim admits an easy inductive argument, and its proof is left as an exercise.

Claim 1. *Every planar triangulation has a normal coloring.*

Let C be a cycle in a planar triangulation T which has been colored normally. A vertex x belonging to C is called a Type i vertex on C if all angles incident with x and interior to C are colored i. When C is the exterior triangle, v_i is a Type i vertex on C.

Claim 2. *If C is a cycle in T, then C contains a Type i vertex for each* $i = 1, 2, 3$.

Proof. Suppose the claim is false. Choose a counterexample C containing the minimum number of elementary triangles. Clearly C is not the boundary of an elementary triangle. Without loss of generality, we may now suppose C does not have a Type 1 vertex.

Suppose that C has two nonconsecutive vertices x and y which are the end points of an edge $e = xy$ interior to C. Then the region bounded by C can be partitioned into regions bounded by cycles C' and C'' having e as a common edge. Now C' and C'' both have a Type 1 vertex. If x is a Type 1 vertex of C' and for C'', then x is a Type 1 vertex for C. An analogous statement holds for y. We conclude that one of x and y is a Type 1 vertex for C' and the other is a Type 1 vertex for C''. Consideration of the two elementary triangles sharing the edge shows this is impossible.

Now let $C = \{x_1, x_2, \ldots, x_s\}$ and let x_i and x_{i+1} be any two consecutive vertices of C, and let z_i be the vertex so that $x_i x_{i+1} z_i$ is an elementary triangle interior to C. Let C_i be the cycle obtained by deleting the edge x_ix_{i+1} and adding the edges x_iz_i and z_ix_{i+1}. Then C_i has a Type 1 vertex because it contains fewer elementary triangles than C. Clearly z_i cannot be a Type 1 vertex on C_i because z_i is an interior vertex of T.

It follows that one of x_i and x_{i+1} is a Type 1 vertex on C_i. If x_i is Type 1 on C_i, then the angle of triangle $x_ix_{i+1}z_i$ incident with x_i must be colored 3; else x_i is Type 1 on C. Thus the angle of $x_ix_{i+1}z_i$ incident with x_{i+1} is colored 1. This implies that x_{i+1} is not Type 1 for C_i. Dually, if x_{i+1} is Type 1 for C_i, then the angle of $x_ix_{i+1}z_i$ incident with x_{i+1} is colored 2, the angle of $x_ix_{i+1}z_i$ incident with x_i is colored 1, and x_i is not Type 1 for C_i.

If some vertex x_{i+1} is Type 1 for both C_i and C_{i+1}, then x_i is Type 1 for C. So either x_i is Type 1 for C_i for $i = 1, 2, \ldots, s$, or x_{i+1} is Type 1 for C_i for $i = 1, 2, \ldots, s$. In the first case, there is no Type 2 vertex on C_1; in the second, there is no Type 3 vertex on C_2. The contradiction completes the proof.

Claim 3. *Let P_i be the binary relation on the set V of vertices* **G** *defined by $(x, y) \in P_i$ if and only if there exists an elementary triangle T having x and y as vertices in which the angle incident at y is colored i. Then the transitive closure $Q_i = \mathrm{tr}(P_i)$ is a partial order on V.*

Proof. It suffices to show Q_i has no directed cycles. This follows from Claim 2 since a directed cycle in Q_i could not have either a Type $i+1$ or a Type $i+2$ vertex.

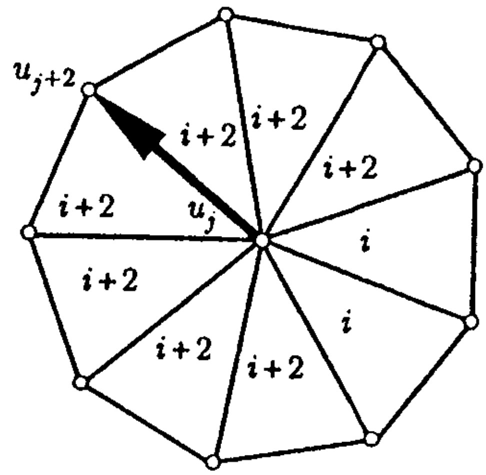
For each $i = 1, 2, 3$, let M_i be a linear extension of Q_i. Then let L_i be any linear extension of P so that:

(1) The restriction of L_i to V is M_i.

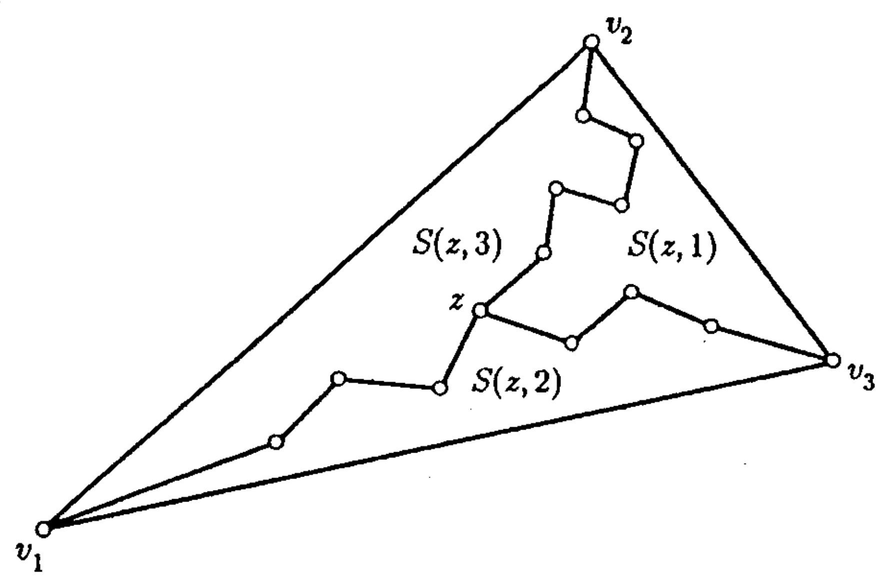
(2) For each $e \in E$, the M_i largest element of V which is less than e in M_i is less than e in P.

Alternately, we may view L_i as being obtained from M_i by inserting the elements of E as low as possible. To complete the proof, it suffices to show that $\mathcal{R} = \{L_1, L_2, L_3\}$ is a realizer of P. To accomplish this, it is enough to show that $\mathcal{R}$ reverses the critical pairs in **P**. However, it is easy to see that the critical pairs have the form (z, e) where $z \in V$, $e = xy \in E$, and z is not an end point of e. It follows that we must find some M_i in which z is over both x and y in M_i. In fact, we will show that there is some i for which $z > x$ in Q_i and $z > y$ in Q_i.

If z is an exterior vertex, say $z = v_i$, then z is the largest element in Q_i. Now suppose z is an interior vertex. Then for each $i = 1, 2, 3$, there is a path $P(z, v_i)$ from z to the i^{th} exterior vertex v_i. The starting point of $P(z, v_i)$ is $u_0 = z$. If u_j has been determined and u_j is an interior vertex, then u_{j+1} is the unique vertex such that the angles at u_j on either side of the edge $u_j u_{j+1}$ are colored $i+1$ and $i+2$.



The paths $P(z, v_1)$, $P(z, v_2)$, and $P(z, v_3)$ are pairwise disjoint (except of course for the point z) and partition T into 3 regions $S(z,1)$, $S(z,2)$, and $S(z,3)$.



If the edge $e = xy$ joins two vertices in the region $S(z,i)$, then z is greater than both x and y in Q_i. This completes the proof. □

3. Convex Polytopes

In this section we generalize Schnyder's characterization of planar graphs and consider the vertices, edges, and faces of a planar map.

We consider a planar map $\mathbf{M}$ as a finite connected planar graph $\mathbf{G} = (V, E)$ together with a plane drawing D of $\mathbf{G}$, i.e., a representation of $\mathbf{G}$ by points and arcs in the plane $\mathbf{R}^2$ in which there are no edge crossings.