2 Generating Functions

Generating Functions: A *generating function* is a formal power series of the form

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

So $f_0, f_1, f_2, \ldots$ is just a sequence of numbers, but we interpret f_k as the coefficient of x^k. Accordingly, we define $f(0) = f_0$.

Exp and Log: We define $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $\log(1+x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k}$

$\mathbb{C}[[\mathbf{x}]]$: We define $\mathbb{C}[[x]]$ to be the collection of formal power series with the variable x and complex coefficients. If $f(x), g(x) \in \mathbb{C}[[x]]$ then $f(x) + g(x)$ and $f(x)g(x)$ are defined in the obvious manner, and this gives $\mathbb{C}[[x]]$ the structure of a commutative ring.

Rule: An expression evaluating to an element of $\mathbb{C}[[x]]$ is only valid if for every $k \in \mathbb{N}$ it is a finite procedure to determine the coefficient of x^k.

Substitution: Let $f(x) = \sum_{k=0}^{\infty} f_k x^k, g(x) \in \mathbb{C}[[x]]$ and assume that $g(0) = 0$. Then we define $f(g(x)) = \sum_{k=0}^{\infty} f_k (g(x))^k$.

Note: In general, and expression of the form $f(x) = \sum_{k=0}^{\infty} f_k x^k$ may be viewed either as a function from a subset of $\mathbb{C}$ to $\mathbb{C}$ or as a generating function. If it converges only when $x = 0$ it is meaningless in the first. Conversely, if it violates the above rule, it is invalid in $\mathbb{C}[[x]]$. So, for instance the equation $\sum_{k=0}^{\infty} \frac{(1+x)^k}{k!} = \exp(1+x) = e \cdot \exp(x) = e \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is valid for functions but not for $\mathbb{C}[[x]]$.

Observation 2.1 *If $f(x) \in \mathbb{C}[[x]]$ then $f(x)$ is invertible if and only if $f(0) \neq 0$.*

Proof: Let $f(x) = \sum_{k=0}^{\infty} f_k x^k$. If $f_0 = 0$ then the constant term of $f(x)g(x)$ is zero for every $g(x)$, so $f(x)$ has no inverse. Conversely, if $f_0 \neq 0$ then we can construct an inverse $g(x) = \sum_{k=0}^{\infty} g_k x^k$ recursively as follows: set $g_0 = \frac{1}{f_0}$ and if we have chosen $g_0, g_1, \ldots, g_j$ so that the first $j+1$ terms of $g(x)f(x)$ are $1 + 0x + 0x^2 \ldots + 0x^j$ then we choose g_{j+1} so that the coefficient of x_{j+1} in $f(x)g(x)$ which is given by

$$\sum_{i=0}^{j+1} f_i g_{j+1-i}$$

is equal to zero (this is possible since $f_0 \neq 0$ and all other terms have already been chosen). $\square$

Example:

$$1 = \exp(x)\exp(-x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This expression is, of course, valid for functions and converges everywhere. Since it does not violate the rule, it follows that this also holds for $\mathbb{C}[[x]]$. Note that this identity is equivalent to: $\sum_{j=0}^{k} (-1)^j \frac{1}{j!(k-j)!} = 0$ if $k > 0$ and is one for $k = 0$. This last equation is recognized as the binomial expansion of $\frac{1}{k!}(1-1)^k$.

The Fibonacci Sequence: This sequence is given by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Setting $f(x) = \sum_{k=0}^{\infty} f_k x^k$ to be the corresponding generating function we have:

$$\begin{aligned} f(x) &= 1 + x + \sum_{k=2}^{\infty} f_k x^k \\ &= 1 + x + \sum_{k=2}^{\infty} f_{k-1} x^k + \sum_{k=2}^{\infty} f_{k-2} x^k \\ &= 1 + x + x \sum_{j=1}^{\infty} f_j x^j + x^2 \sum_{j=0}^{\infty} f_j x^j \\ &= 1 + x + x(f(x) - 1) + x^2 f(x) \end{aligned}$$

Rearranging, we find

$$f(x) = \frac{-1}{x^2 + x - 1}$$

so $f(x)$ is -1 times the inverse of the generating function $-1 + x + x^2$ and to find $f(x)$ we simply need to invert $-1 + x + x^2$. To do this, we note that this polynomial is equal to $(x-\alpha)(x-\beta)$ where $\alpha = \frac{-1-\sqrt{5}}{2}$ and $\beta = \frac{-1+\sqrt{5}}{2}$ so by partial fractions

$$\begin{aligned} f(x) &= \frac{-1}{x^2 + x - 1} \\ &= -\frac{5^{-1/2}}{\alpha - x} + \frac{5^{-1/2}}{\beta - x} \\ &= -5^{-1/2}\alpha^{-1} \sum_{k=0}^{\infty} (\alpha^{-1})^k x^k + 5^{1/2}\beta^{-1} \sum_{k=0}^{\infty} (\beta^{-1})^k x^k \qquad \square \end{aligned}$$

Making Change: Observe that for a positive integer n, the number of distinct collections of coins which have total value n is given by the coefficient of x^n in the power series

$$\left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} x^{5k}\right)\left(\sum_{k=0}^{\infty} x^{10k}\right)\left(\sum_{k=0}^{\infty} x^{25k}\right) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right)\left(\frac{1}{1-x^{25}}\right)$$

To see this, observe that each term which contributes to x^n in the above product arises by choosing a power of x, say k_1 from the first function, a power of x^5, say k_2 from the second, a power of x^{10}, say k_3 from the third, and a power of x^{25}, say k_4 from the fourth in such a way that $k_1 + 5k_2 + 10k_3 + 25k_4 = n$.

Theorem 2.2 *Fix a finite field* $\mathbb{F}_q$ *and let* N_d *be the number of monic irreducible polynomials of degree* d *over* $\mathbb{F}$. *Then*

$$q^n = \sum_{d|n} dN_d$$

Proof: First note that the number of monic polynomials of degree n is precisely q^n, so the generating series given by this sequence is precisely

$$1 + qx + (qx)^2 + \ldots = \frac{1}{1-qx} \tag{1}$$

Now, enumerate the monic irreducible polynomials as $f_1(x), f_2(x), \ldots$ where $deg(f_i) \leq deg(f_{i+1})$ for all $i \geq 1$ and let $d_i = deg(f_i)$. Every monic polynomial of degree n has a unique factorization as $f_1(x)^{k_1} f_2(x)^{k_2} \ldots$ where $\sum_{i=1}^{\infty} k_1 = n$, so (as in the previous example) the number of monic polynomials of degree n is equal to the coefficient of x^n in

$$\prod_{i=1}^{\infty} (1 + x^{d_i} + x^{2d_i} + \ldots) = \prod_{i=1}^{\infty} \frac{1}{1-x^{d_i}} = \prod_{d=1}^{\infty} \left(\frac{1}{1-x^d}\right)^{N_d}$$

Now, equating the expressions we have for the generating function of monic polynomials found in (1) and above and then taking formal logarithms gives us the following: (here we use $\log \frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$)

$$\sum_{n=1}^{\infty} \frac{(qx)^n}{n} = \sum_{d=1}^{\infty} N_d \sum_{j=1}^{\infty} \frac{x^{jd}}{j}$$

Comparing the coefficient of x^n in each of these series gives us (here we note that the only terms on the right which contribute are those for which $d|n$ and in this case we have $j = n/d$):

$$\frac{q^n}{n} = \sum_{d|n} N_d \frac{1}{n/d}$$

From which we get $q^n = \sum_{d|n} dN_d$ as required. $\square$

Exponential Generating Functions: A formal power series of the form $f(x) = \sum_{k=0}^{\infty} f_k \frac{x^k}{k!}$ is called an *exponential generating function* for the sequence $f_0, f_1, \ldots$.

Differentiation: For $f(x) \in \mathbb{C}[[x]]$ we define $f'(x)$ in the obvious manner. This gives $\mathbb{C}[[x]]$ the structure of a formal calculus.

Derangements: For every $i \in \mathbb{N}$ let d_i denote the number of derangements of $[n]$ (permutations $\pi \in S_n$ for which $\pi(i) \neq i$ for all $i \in [n]$). Let $n \geq 2$ let $\pi \in S_n$ be a derangement and consider its cycle representation. If there is a cycle of the form (n, j), then removing this term results in a derangement of $[n-1] \setminus \{j\}$ (and there are exactly d_{n-2} of these) so the total number of derangements with such a term is precisely $(n-1)d_{n-2}$. If there is no cycle of this form, then we may remove n from the cycle containing it to get a derangement of $[n-1]$. Since every derangement of $[n-1]$ gives rise to $n-1$ derangements of $[n]$ in this manner (as n may be inserted in any of $n-1$ places) we find that

$$d_n = (n-1)(d_{n-1} + d_{n-2})$$

Now, set $D(x) = \sum_{k=0}^{\infty} d_k \frac{x^k}{k!}$. Then by elementary manipulations, we find

$$(1-x)D'(x) = xD(x)$$

From which it follows that $D(x) = \exp(-x)/(1-x)$ so

$$d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$