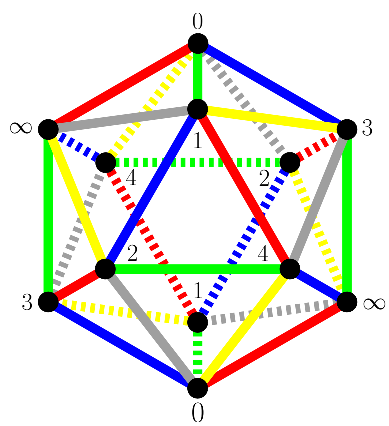
Galois' Exceptional Actions

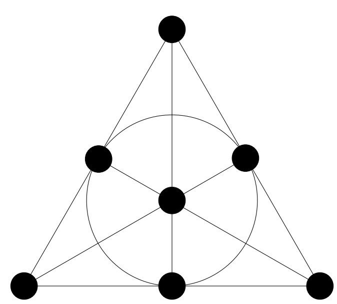
Theorem 1 (Galois) *The group $PSL(2,q)$ has a faithful action on q points only when $q = 5, 7, 11$.*

$\mathbf{PSL(2,5)}$: The rotational symmetry group of the icosahedron is isomorphic to A_5 and to $PSL(2,5)$ as evidenced by the figure below. To see that this group is isomorphic to A_5, note that each rotational symmetry corresponds uniquely to an even permutation of the five colour classes. To see that it is isomorphic to $PSL(2,5)$, note that the rotational symmetries acting on the diagonals - or equivalently the opposite pairs of vertices (labelled by $0,1,2,3,4,\infty$) give the usual action of $PSL(2,5)$ on the projective line over $\mathbb{F}_5$.



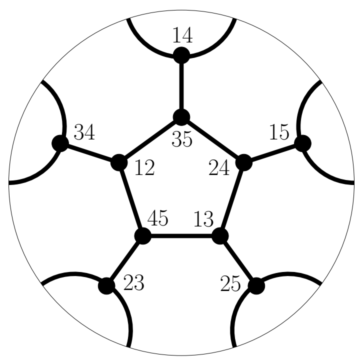
Note: Since $PSL(2,5) \cong A_5$ we see that its natural action on 5 points has a 5-cycle and the stabilizer of a point is isomorphic to A_4.

$\mathbf{PSL(2,7)}$: The automorphism group of the Fano plane is isomorphic to both $PSL(2,7)$ and to $GL(3,2)$. The latter isomorphism is quite immediate as the Fano Plane is $PG(2,2)$ which is naturally acted on by $GL(3,2)$.



Note: The Fano Plane can also be constructed by taking the point set to be $\mathbb{F}_7$ and the blocks to be all sets of the form $i + \mathbb{F}_7^\square$ for $i \in \mathbb{F}_7$. It follows that our action of $PSL(2,7)$ on seven points has a 7-cycle. The stabilizer of a point is isomorphic to S_4.

$\mathbf{PSL(2,11)}$: There is a unique 2-$(11,5,2)$ design which can be constructed as follows. Take an embedding of the Petersen graph in the projective plane as shown in the figure below, where opposite points of the disc are identified. Let $\mathcal{V}$ be the vertices of this graph together with a new point ∞. Now for each face we let these five vertices form a block, and for each of the five independent sets of size four (in the figure the i^{th} independent set $i = 1 \ldots 5$ is the set of vertices whose label contains i) we add a block containing these points and ∞. The group $PSL(2,11)$ is the automorphism group of this design.



Note: This design can also be constructed by taking the points to be $\mathbb{F}_{11}$ and the blocks to be all sets of the form $i + \mathbb{F}_{11}^\square$ for $i \in \mathbb{F}_{11}$. It follows from this that our action of $PSL(2,11)$ on eleven points has an 11-cycle. The stabilizer of a point is isomorphic to the symmetry group of our embedded Petersen, which is A_5.

Coincidence? The point stabilizers of the exceptional actions of $PSL(2,5)$, $PSL(2,7)$, and $PSL(2,11)$ are A_4, S_4, and A_5 which are precisely the rotational symmetry groups of the Platonic solids.