Strong Orientations of Plane Graphs with Bounded Stretch Factor

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Abstract

We study the problem of orienting a subset of edges of given plane graph such that the resulting subdigraph is strongly connected and spans all vertices of the graph. We are interested in orientations with minimum number of arcs and such that they produce a digraph with bounded stretch factor. Such orientations have applications into the problem of establishing strongly connected sensor network when sensors are equipped with directional antennae.

We present three constructions for such orientations. Let $G = (V, E)$ be a connected plane graph without cut edges and let $\Phi(G)$ be the degree of largest face in G. Our constructions are based on a face coloring, say with λ colors.

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First construction gives a strong orientation with at most $\left(2 - \frac{4\lambda-6}{\lambda(\lambda-1)}\right)|E|$ arcs and stretch factor at most $\Phi(G) - 1$. The second construction gives a strong orientation with at most $|E|$ arcs and stretch factor at most $(\Phi(G) - 1)^{\lceil \frac{\lambda+1}{2} \rceil}$. The third construction can be applied to plane graphs which are 3-edge connected. It uses a particular 6-face coloring and for any integer $k \geq 1$ produces a strong orientation with at most $(1 - \frac{k}{10(k+1)})|E|$ arcs and stretch factor at most $\Phi^2(G)(\Phi(G) - 1)^{2k+4}$.

Key words and phrases: Digraph, Directional Antennae, Plane Graph, Sensors, Cut Edges, Spanner, Stretch Factor, Strongly Connected.

1 Introduction

Directional antennae are widely being used in wireless networks not only for reducing energy consumption and interference, but also for improving routing efficiency and security. Sensors rely on the use of antennae to configure and operate an ad hoc network. Essentially, two types of antennae are being used: *Omnidirectional* antennae which transmit the signal in all directions in the plane and *directional* antennae which can transmit the signal towards a specific direction. Omnidirectional antennae usually incur more interference than directional antennae thus hampering nodes from receiving data from other transmitters and causing overall performance degradation of the sensor network. Sensor networks using directional antennae not only can have extended life-time since the consumption of energy in each antenna is proportional to the area covered by the transmitting antennae, but also using a small antenna spread prevents unwanted nodes from listening to the communication and therefore, improving the throughput and security of the network. Hence, it is desirable to reduce not only the range, but also the angle of an antenna.

There has been some recent research concerning the advantages of using directional antennae. For example, Gupta et al. [7] have shown that when n omnidirectional antennae are being used in an area of a unit square, the throughput per node is at most $O(W/\sqrt{n})$, where each antenna can transmit W bits per second over the common channels, regardless of the sensor placement. This can be contrasted with Yi et al. [13] which show that directional antennae provide an improvement on the throughput capacity by a factor of $2\pi/\sqrt{\alpha\beta}$, where α is the angle of transmission and β is the angle of reception. In fact, when α and β go to zero, the wireless network behaves like a wired network from the throughput point of view. Similarly, Kranakis et al. [9] studied the energy consumption of networks of omnidirectional antennae and compared it to the consumption of networks of directional antennae.

Motivated from these issues, in this paper we study the problem of directing edges of an undirected (connected) plane graph in such a way that the resulting digraph spans all vertices, is strongly connected, has bounded stretch factor, and the number of arcs employed is minimized. Note that if the undirected graph is hamiltonian then a solution is to orient edges along a Hamilton cycle. This yields an orientation that is strongly connected and has the minimum possible number of arcs. However the stretch factor of such orientation is unbounded. On the other hand, one can orient every edge of an undirected graph in two oposite directions. This will result in an orientation that is strongly connected with stretch factor equal to one. However the number of arcs in such an orientation is largest possible. Hence we are looking for tradeoffs between these two approaches.

1.1 Notation and Preliminaries

A set of sensors with omnidirectional antennae is modeled as an (undirected) geometric graph whose vertices are points in the plane and edges are straight line segments representing the connectivity between two sensors. A set of sensors with directional antennae is modeled as a directed geometric graph (digraph) where the direction on an arc represents the direction of the corresponding communication link. Geometric graphs are always associated with straight line planar embeddings and so we will consider them as plane graphs (digraphs) with all edges straight lines, and will speak of their set of vertices V, edges E, and faces F, respectively. Our graphs (digraphs) will not have loops and/or multiple edges (arcs). Given two integers $a < b$, let $[a, b] = \{a, a+1, \dots, b\}$ denote the integer interval. A (face) λ-coloring $\Lambda : F \to [1, \lambda]$ of a plane graph $G(V, E, F)$ is an assignment Λ of λ colors to faces of G such that adjacent faces, i.e. faces sharing a common edge, are assigned distinct colors.

Let G be a graph. An orientation $\vec{G}$ of G is a digraph obtained from G by orienting every edge of G in at least one direction. As usual, we denote with (u, v) the arc from u to v, whereas $\{u, v\}$ denotes an undirected edge between u and v. Let $d^+(u)$ denote the out-degree of u in $\vec{G}$. Similarly, by $d(f)$ we denote the degree (the number of edges) of the face f in a plane graph (digraph) G. In both cases, if an ambiguity can occur, we expand the notation by a subscript representing the corresponding graph, e.g. $d_{\vec{G}}(u)$ denotes the out-degree of u in the digraph $\vec{G}$. Finally let $\Phi(G)$ be the maximum degree of a face in G, i.e. $\Phi(G) = \max_{f \in F} d(f)$.

The stretch factor or spanning ratio of a strongly connected orientation $\vec{G}$ is the minimum value t such that for every ordered pair of vertices u and v and for every path from u to v in G there exists a directed path from u to v in $\vec{G}$ of length at most t times the length of the original path.

1.2 Related Work

Caragiannis et al. [3] were the first to propose the problem of orienting the antennae of a set of sensors in the plane and compared the range used to the maximum edge length of the minimum spanning tree on the set of sensors. They proposed a polynomial time algorithm in which the sensors use the optimal possible range to maintain connectivity in order to construct a strongly connected graph when the antennae spread is at least $8\pi/5$. In addition, they studied the case when the antennae spread α is in the interval $[\pi, 8\pi/5)$ and gave an algorithm which extends the antenna range to at most $2\sin(\pi - \alpha/2)$ times the minimum range required so as to maintain connectivity. They also showed that if the antenna spread is at most $2\pi/3$ the problem of constructing a strongly connected graph is NP-hard. In fact, when the angles of the antennae are equal to 0 this last problem is equivalent to the bottleneck traveling salesman problem [11] for which an approximation with radius 2 times the optimal is given in [11].

Bhattacharya et al. [1] extend the work in [3] and give results for more than one antenna per sensor. A more comprehensive study is provided by Dobrev et al. [5]. They consider the previously mentioned model of Caragiannis et al. [3] in order to study the optimal antennae range required when sensors are equipped with more than one antenna having spread 0. They show that the required range is $\sqrt{3}$ times the optimal for two antennae, $\sqrt{2}$ times the optimal for three antennae and $2\sin(\pi/5)$ times the optimal for four antennae.The problem considered in the present paper differs from the problems studied in [3], [1] and [5] in that we do not alter (increase) the sensor range, rather we work with given undirected graph (unit disk graph or its planar spanner).

Similar problem that has been addressed in the literature is one that studies connectivity requirements on undirected graph that will guarantee highest edge connectivity of its orientation, c.f. [6] and [10].

1.3 Contributions

Let $G = (V, E)$ be a connected plane graph without cut edges and let Λ be a face coloring, say with λ colors. We present three polynomial constructions for orientations of G.

First construction (presented in Section 2) gives a strong orientation with at most $\left(2 - \frac{4\lambda-6}{\lambda(\lambda-1)}\right)|E|$ arcs and the stretch factor at most $\Phi(G) - 1$. The second construction (presented in Section 3) gives a strong orientation with at most $|E|$ arcs and the stretch factor at most $(\Phi(G)-1)^{\lceil \frac{\lambda+1}{2} \rceil}$. The third construction (presented in Section 4) can be applied to plane graphs which are 3-edge connected. It uses a particular 6-face coloring and for any integer $k \geq 1$ produces a strong orientation with at most $(1 - \frac{k}{10(k+1)})|E|$ arcs and the stretch factor at most $\Phi^2(G)(\Phi(G)-1)^{2k+4}$.

2 Orientations with more that $|E|$ arcs

Let $G(V, E, F)$ be a simple plane geometric graph. We want to orient edges in E so that the resulting digraph is strongly connected. A trivial algorithm is to orient each edge in E in both directions. In this case, the number of arcs is $2|E|$ and the stretch factor is 1. In this section we prove that it is possible to orient less than $2|E|$ edges of G and still maintain bounded stretch factor. Our approach is based on a λ-coloring of faces in F, for some integer λ. The idea of employing face coloring was used in [14] to construct directed cycles. Intuitively we give directions to edges depending on the color of their incident faces.

Theorem 1 *Let $G(V, E, F)$ be a plane geometric graph which is 2-edge connected. Suppose G has a face λ-coloring for some integer λ. There exists a strongly connected orientation $\vec{G}$ with at most*

$$\left(2 - \frac{4\lambda - 6}{\lambda(\lambda-1)}\right) \cdot |E| \tag{1}$$

arcs, so that its stretch factor is at most $\Phi(G) - 1$.

Before giving the proof, we introduce some useful ideas and preliminary results that will be required.

Consider a plane geometric graph $G(V, E, F)$ and a face λ-coloring Λ of G with colors $\{1, 2, \ldots, \lambda\}$. Let $\vec{G}$ be the orientation resulting from giving two opposite directions to each edge in E. For each arc (u, v), we define $L_{u,v}$ as the face which is incident to $\{u, v\}$ on the left of (u, v), and similarly $R_{u,v}$ as the face which is incident to $\{u, v\}$ on the right of (u, v). Observe that for given embedding of G, $L_{u,v}$ and $R_{u,v}$ are well defined. Since G has no cut edges, $L_{u,v} \neq R_{u,v}$. This will be always assumed in the proofs below without specifically recalling the reason again. We classify arcs according to the colors of their incident faces. Let $E_{i,j}$ be the set of arcs (u, v) in $\vec{G}$ such that $\Lambda(L_{u,v}) = i$ and $\Lambda(R_{u,v}) = j$. It is easy to see that each arc is exactly in one such set. Hence, the following lemma is evident and can be given without proof.

Lemma 1 *For any face λ-coloring of a plane geometric graph G,*

$$\sum_{i=1}^{\lambda} \sum_{j=1, j \neq i}^{\lambda} |E_{i,j}| = 2|E|.$$

For any of $\lambda(\lambda - 1)$ ordered pairs of two distinct colors a and b in the coloring Λ, we define the digraph $D(G; a, b)$ as follows: The vertex set of the digraph D is V and the arc set of D is

$$\bigcup_{i \in [1,\lambda]] \setminus \{b\}, j \in [1,\lambda] \setminus \{a\}} E_{i,j}.$$

Along with this definition, for $i \neq b$, $j \neq a$, and $i \neq j$, we say that $E_{i,j}$ is in $D(G; a, b)$. Next consider the following characteristic function

$$\chi_{a,b}(E_{i,j}) = \begin{cases} 1 & \text{if } E_{i,j} \text{ is in } D(G; a, b), \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that every set $E_{i,j}$ is in exactly $\lambda^2 - 3\lambda + 3$ different digraphs $D(G; a, b)$ for some $a \neq b$.

Lemma 2 *For any face λ-coloring of a plane geometric graph G,*

$$\sum_{a=1}^{\lambda} \sum_{b=1, b \neq a}^{\lambda} \chi_{a,b}(E_{i,j}) = \lambda^2 - 3\lambda + 3.$$

Proof Let $i, j \in [1, \lambda]$, $i \neq j$ be fixed. For any two distinct colors a and b of the λ-coloring of G, $\chi_{a,b}(E_{i,j}) = 1$ only if either $i = a$, or $j = b$, or i and j are different from a and b. There are $(\lambda - 1) + (\lambda - 2) + (\lambda - 2)(\lambda - 3)$ such colorings. The lemma follows by simple counting. □

The following lemma gives a key property of the digraph $D(G; a, b)$.

Lemma 3 *Given a face λ-coloring of a plane geometric graph G with no cut edges, and the corresponding digraph $D(G; a, b)$. Every face of $D(G; a, b)$, which has color a, constitutes a counter clockwise directed cycle, and every face which has color b, constitutes a clockwise directed cycle. All arcs on such cycles are unidirectional. Moreover, each arc of $D(G; a, b)$ incident to faces having colors different from either a or b is bidirectional.*

Proof Let G be a plane geometric graph with a face λ-coloring Λ with colors a, b and $\lambda - 2$ other colors. Consider $D(G; a, b)$. The sets $E_{a,x}$ are in $D(G; a, b)$ for each color $x \neq a$. Let f be a face and let $\{u, v\}$ be an edge of f so that $L_{u,v} = f$. Let f' be the other face incident to $\{u, v\}$; hence $R_{u,v} = f'$.

Since G has no cut edges, $f \neq f'$, and since $\Lambda(f') \neq a$, the arc $(u, v) \in \bigcup_{x \neq a} E_{a,x}$ and hence the arc (u, v) is in $D(G; a, b)$. Since $\{u, v\}$ was an arbitrary edge of f, f will induce a counter clockwise cycle in $D(G; a, b)$. The fact that every face which has color b induces a clockwise cycle in $D(G; a, b)$ is similar.

Finally consider an edge $\{u, v\}$ such that $\Lambda(L_{u,v}) \neq a, b$ and $\Lambda(R_{u,v}) \neq a, b$. Hence $(u, v) \in E_{\lambda,d}$ which is in $D(G; a, b)$ and similarly $(v, u) \in E_{d,\lambda}$ which is also in $D(G; a, b)$. This proves the lemma. $\square$

We are ready to prove Theorem 1.

Proof [Theorem 1] Let G be a plane geometric graph having no cut edges. Let Λ be a face λ-coloring of G with colors a, b, and other $\lambda - 2$ colors. Suppose colors a and b are such that the corresponding digraph $D(G; a, b)$ has the minimum number of arcs. Consider $\overline{A}$ the average number of arcs in all digraphs arising from Λ. Thus,

$$\overline{A} = \frac{1}{\lambda(\lambda-1)} \sum_{a=1}^{\lambda} \sum_{b=1, b\neq a}^{\lambda} |D(G; a, b)|, \text{where}$$

$$|D(G; a, b)| = \sum_{i=1}^{\lambda} \sum_{j=1, j\neq i}^{\lambda} \chi_{a,b}(E_{i,j})|E_{i,j}|.$$

By Lemma 1 and Lemma 2,

$$\begin{aligned}
\overline{A} &= \frac{1}{\lambda(\lambda-1)} \sum_{a=1}^{\lambda} \sum_{b=1, b\neq a}^{\lambda} \sum_{i=1}^{\lambda} \sum_{j=1, j\neq i}^{\lambda} \chi_{a,b}(E_{i,j})|E_{i,j}| \\
&= \frac{1}{\lambda(\lambda-1)} \sum_{i=1}^{\lambda} \sum_{j=1, j\neq i}^{\lambda} (\lambda^2 - 3\lambda + 3)|E_{i,j}| \\
&= \frac{2(\lambda^2 - 3\lambda + 3)}{\lambda(\lambda-1)} |E| \\
&= \left(2 - \frac{4\lambda - 6}{\lambda(\lambda-1)}\right) |E|.
\end{aligned}$$

Hence $D(G; a, b)$ has at most the desired number of arcs.

To prove the strong connectivity of $D(G;a,b)$, consider any path, say $u = u_0, u_1, \ldots, u_n = v$, in the graph G from u to v. We prove that there exists a directed path from u to v in $D(G;a,b)$. It is enough to prove that for all i there is always a directed path from u_i to u_{i+1} for any edge $\{u_i, u_{i+1}\}$ of the above path. We distinguish several cases.

- Case 1. $\Lambda(L_{u_i,u_{i+1}}) = a$. Then $(u_i, u_{i+1}) \in E_{a,\omega}$ where $\omega = \Lambda(R_{u_i,u_{i+1}})$. Since $E_{a,\omega}$ is in $D(G;a,b)$, the arc (u_i, u_{i+1}) is in $D(G;a,b)$. Moreover, the stretch factor of $\{u_i, u_{i+1}\}$ is one.

- Case 2. $\Lambda(L_{u_i,u_{i+1}}) = b$. Hence, (u_i, u_{i+1}) is not in $D(G;a,b)$. However, by Lemma 3, the face $L_{u_i,u_{i+1}} = R_{u_{i+1},u_i}$ constitutes a clockwise directed cycle, and therefore, a directed path from u_i to u_{i+1}. It is easy to see that the stretch factor of $\{u_i, u_{i+1}\}$ is not more than the size of the face $L_{u_i,u_{i+1}}$ minus one, which is at most $\Phi(G) - 1$.

- Case 3. $\Lambda(L_{u_i,u_{i+1}}) \neq a, b$. Suppose $\Lambda(L_{u_i,u_{i+1}}) = c$. Three cases can occur.

  - $\Lambda(R_{u_i,u_{i+1}}) = a$. Hence, (u_i, u_{i+1}) is not in $D(G;a,b)$. However, by Lemma 3, there exists a counter clockwise directed cycle around face $R_{u_i,u_{i+1}} = L_{u_{i+1},u_i}$, and consequently a directed path from u_i to u_{i+1}. The stretch factor is at most the size of face $R_{u_i,u_{i+1}}$ minus one, which is at most $\Phi(G) - 1$.
  - $\Lambda(R_{u_i,u_{i+1}}) = b$. By Lemma 3, there exists a clockwise directed cycle around face $R_{u_i,u_{i+1}}$. This cycle contains (u_i, u_{i+1}), and in addition the stretch factor of $\{u_i, u_{i+1}\}$ is one.
  - $\Lambda(R_{u_i,u_{i+1}}) = d \neq a, b, c$. By the construction, $D(G;a,b)$ has both arcs (u_i, u_{i+1}) and (u_{i+1}, u_i). Again, the stretch factor of $\{u_i, u_{i+1}\}$ is one.

This proves the theorem. $\square$

As indicated in Theorem 1 the number of arcs in the orientation depends on the number λ of colors. Thus, for specific values of λ we have the following table of values:

λ	3	4	5	6	7
$2 - \frac{4\lambda-6}{\lambda(\lambda-1)}$	1	$\frac{7}{6}$	$\frac{13}{10}$	$\frac{7}{5}$	$\frac{31}{21}$

Regarding the complexity of the algorithm, this depends on the number λ of colors being used. For example, computing a 4-coloring can be done in $O(n^2)$ [12]. Finding the digraph with minimum number of arcs among the twelve possible digraphs can be done in linear time. Therefore, for $\lambda = 4$, the orientation can be computed in

$O(n^2)$. For $\lambda = 5$ a 5-coloring can be found in $O(n)$ time. For geometric plane subs-graphs of unit disk graphs and location aware nodes there is a local 7-coloring (see [4]). For more information on colorings the reader is advised to look at [8]. We also have the following corollary.

Corollary 1 *Let $G = (V, E, F)$ be a geometric plane triangulation. There exists a strongly connected orientation $\vec{G}$ with at most $7(|V| - 2)/2$ arcs and stretch factor of 2.*

3 Orientations with $|E|$ arcs

Theorem 1 shows that every geometric plane graph G without cut edges has a strong orientation with bounded stretch factor and at most $\left(2 - \frac{4\lambda-6}{\lambda(\lambda-1)}\right) \cdot |E|$ arcs. In this section we show that one can orient every edge in exactly one direction only and still obtain a strong orientation. However the stretch factor will increase.

Consider a geometric plane graph $G(V, E, F)$ having no cut edges and a face λ-coloring Λ of G with colors $[1, \lambda]$. Let $\vec{G}$ be the orientation assigning two opposite directions to each edge of E. Let $E_{i,j}$ be the set of arcs (u, v) in $\vec{G}$ such that $\Lambda(L_{u,v}) = i$ and $\Lambda(R_{u,v}) = j$. Recall that since G has no cut edges, $L_{u,v} \neq R_{u,v}$. Clearly, these sets are pairwise disjoint. We define the digraph $D(G; \Lambda)$ as follows: The vertex set of the digraph $D(G; \Lambda)$ is V and the arc set of $D(G; \Lambda)$ is $\bigcup_{i<j\leq\lambda} E_{i,j}$.

It is not difficult to observe that in $D(G; \Lambda)$ exactly one direction is assigned to every edge of G.

Theorem 2 *Let $G(V, E, F)$ be a geometric plane graph which is 2-edge connected. For any face λ-coloring Λ of G, the digraph $D(G; \Lambda)$ is strongly connected, has exactly $|E|$ arcs, and its stretch factor is at most $(\Phi(G) - 1)^{\lceil \frac{\lambda+1}{2} \rceil}$.*

Proof We already observed above that $D(G; \Lambda)$ has $|E|$ arcs. We prove the following two statements.

1. We first prove by induction on k that if $\{u, v\} \in E$ so that $\Lambda(L_{,v}) = k$ then if (u, v) is in $D(G; \Lambda)$ then there is also a directed path from v to u in $D(G; \Lambda)$ of length at most $(\Phi(G) - 1)^k$ such that every arc on this path is incident to a face of color at most k.

2. Second we prove that for every k if $\{u, v\} \in E$ so that $\Lambda(R_{u,v}) = k$ then if (v, u) is in $D(G; \Lambda)$ then there is also a directed path from u to v in $D(G; \Lambda)$ of length at most $(\Phi(G) - 1)^{\lambda-k+1}$ such that every arc on this path is incident to a face of color at least k.

The theorem follows easily. Indeed, let $\{u,v\} \in E$ so that $\Lambda(L_{u,v}) < \Lambda(R_{u,v})$. The arc (u,v) constitutes the required directed path from u to v. We exhibit required directed path from v to u as follows. If $\Lambda(L_{u,v}) \leq \lceil \frac{\lambda}{2} \rceil$, then the directed path from v to u exists by first statement. If $\Lambda(L_{u,v}) > \lceil \frac{\lambda}{2} \rceil$, then since $L_{u,v} = R_{v,u}$, $\Lambda(R_{v,u}) > \lceil \frac{\lambda}{2} \rceil$ and since $(u,v) \in E$, the required directed path from v to u exists by the second statement above.

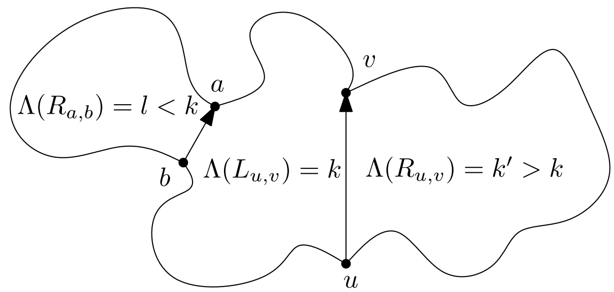

Figure 1: The figure shows how to find a directed path from v to u when (u,v) is in $D(G;\Lambda)$. If $\{a,b\}$ is any edge incident to $L_{u,v}$ then by inductive hypothesis there is a path from a to b of length at most $(\Phi - 1)^l$.

Next we give the proof of the statement 1 above. Base step is $\Lambda(L_{u,v}) = 1$. Hence, the other face incident to $\{u,v\}$ has color $j > 1$. Therefore, $(u,v) \in E_{1,j}$ which is in $D(G;\Lambda)$ by definition. We have the same conclusion for any other arc of the face $L_{u,v}$ and hence this face will induce a directed cycle in $D(G;\Lambda)$, which provides a desired directed path from v to u. The length of this path is obviously at most $\Phi(G) - 1$. Also every arc of this path is obviously incident to a face of color 1. This proves the base case.

In the inductive step we assume the statement is true for all $l \leq k-1$. We prove it for k. Assume $\Lambda(L_{u,v}) = k$. If $k < \Lambda(R_{u,v}) = k'$, then $(u,v) \in E_{k,k'}$ which is in $D(G;\Lambda)$ by definition. To construct a directed path from v to u, consider the face $L_{u,v}$ and any edge $\{a,b\}$ incident to this face so that $L_{a,b} = L_{u,v}$. If $\Lambda(L_{a,b}) < \Lambda(R_{a,b})$, the arc (a,b) is in $D(G;\Lambda)$, and $\Lambda(L_{a,b}) = k \leq k$ as required. Otherwise the arc (b,a) is in $D(G;\Lambda)$ and also $l = \Lambda(L_{b,a}) = \Lambda(R_{a,b}) < \Lambda(L_{a,b}) = k$ (see Figure 3). Thus, by inductive hypothesis, there is a directed path from a to b in $D(G;\Lambda)$ of length at most $(\Phi(G)-1)^l$ such that every arc of this path is incident to a face of color at most $\Lambda(L_{b,a}) = l \leq k$. At most $(\Phi(G) - 1)$ arcs of $L_{u,v}$ will be replaced in this way, so the length of the desired path from v to u is at most $(\Phi(G) - 1)^k$. If $k > \Lambda(R_{u,v}) = k'$, then (u,v) is not in $D(G;\Lambda)$.

Finally we give the proof of the statement 2 above. Base step is $\Lambda(R_{u,v}) = \lambda$ which is trivially true since (v, u) is not in $D(G; \Lambda)$. In the inductive step we assume the statement is true for all $l \geq \lambda - k + 1$. We prove it for $\lambda - k$. Assume $\Lambda(R_{u,v}) = \lambda - k$. If $\lambda - k > \Lambda(L_{u,v}) = k'$, then $(u, v) \in E_{k',\lambda-k}$ which is in $D(G; \Lambda)$ by definition and hence (v, u) is not in $D(G; \lambda)$. Hence suppose $\lambda - k < \Lambda(L_{u,v}) = k'$. Hence $(v, u) \in E$. To construct a directed path from u to v, consider the face $L_{u,v}$ and any edge $\{a, b\}$ incident to this face so that $R_{a,b} = L_{u,v}$. If $\Lambda(L_{a,b}) < \Lambda(R_{a,b})$, the arc (a, b) is in $D(G; \Lambda)$, and $\Lambda(L_{a,b}) = k' \geq \lambda - k$ as required. Otherwise the arc (b, a) is in $D(G; \Lambda)$ and also $\Lambda(R_{a,b}) = k'$. Thus, by inductive hypothesis, there is a directed path from a to b in $D(G; \Lambda)$ of length at most $(\Phi(G) - 1)^{\lambda - k' + 1}$ such that every arc of this path is incident to a face of color at lest $\Lambda(R_{a,b}) = k' > \lambda - k$. At most $(\Phi(G) - 1)$ arcs of $L_{u,v}$ will be replaced in this way, so the length of the desired path from u to v is at most $(\Phi(G) - 1)^{\lambda - k' + 1 + 1} \leq (\Phi(G) - 1)^{k+1}$, since $k' \geq \lambda - k + 1$. This proves the theorem. □

As a corollary we obtain the following result on triangulations.

Corollary 2 *Let $G(V, E, F)$ be a geometric plane triangulation, and let Λ be its face 4-coloring. The digraph $D(G; \Lambda)$ is strongly connected, has exactly $|E|$ arcs, and its stretch factor is at most 8.*

4 Orientations with less that $|E|$ arcs

By considering more sophisticated face colorings, we can decrease the number of arcs below $|E|$ in a strong orientation and still maintain a bounded stretch factor. Define

$$D'(G; \Lambda) = D(G; \Lambda) - E_{\lceil \frac{\lambda-1}{2} \rceil, \lceil \frac{\lambda+1}{2} \rceil}.$$

We use the following result about acyclic coloring of plane graphs. A (proper vertex) coloring is acyclic if every subgraph induced by any two colors is acyclic.

Theorem 3 *[2] Every plane graph has an acyclic coloring with 5 colors.*

Lemma 4 *Let $T = (V, E)$ be a forest and $k \geq 1$ an integer. There exists a set of vertices $S \subseteq V$ such that the subgraph of T induced by S is a forest of trees of diameter at most $4k$ and has at least $\frac{k}{k+1}|E|$ arcs.*

Proof In this proof all indices will be considered modulo $2k+2$. Root every component of T at any vertex and consider the partition of V into k sets $V_0, V_1, \ldots, V_{2k+1}$ such that the set

$$V_\ell = \{x \in V \ : \ \text{distance of } x \text{ from the root of its component is } \ell \bmod 2k + 2\}.$$

Now consider the following k forests of trees of diameter at most $4k$. For $m = 0, 1, \ldots, k-1$, let $G^m = (V^m, E^m)$ where

$$\begin{aligned} V^m &= V_{0-2m} \cup V_{1-2m} \cup \cdots \cup V_{2k-2m}\,, \\ E^m &= \{\{x,y\} \in E \,:\, x \in \cup_{i=0-2m}^{2k-1-2m} V_i,\; y \in \cup_{i=1-2m}^{2k-2m} V_i\}. \end{aligned}$$

It is not difficult to see that every G^m is, in fact, induced subgraph of T. If one of the graphs G^m has at least $\frac{k}{k+1}|E|$ arcs, we are done. On the other hand, only edges of T that are not included in G^m for given m are edges $\{x, y\}$ such that $x \in V_{2k-2m}$ and $y \in V_{2k+1-2m}$ and edges $\{x, y\}$ such that $x \in V_{2k+1-2m}$ and $y \in V_{2k+2-2m}$, i.e. edges of stars centered at vertices in $V_{2k+1-2m}$. Since G^m has less than $\frac{k}{k+1}|E|$ edges, there is at least $\frac{1}{k+1}|E|$ such edges. This must be true for all $m = 0, 1, \ldots, k-1$, and these edge sets are pairwise disjoint for distinct values of m. Hence the graph $G^k = (V^k, E^k)$ such that

$$\begin{aligned} V^k &= V_2 \cup V_3 \cup \cdots \cup V_{2k+2}, \\ E^k &= \{\{x,y\} \in E \,:\, x \in \cup_{i=2}^{2k+1} V_i, y \in \cup_{i=3}^{2k+2} V_i\} \end{aligned}$$

is the forest of trees of diameter at most $4k$, is induced in T and has at least $\frac{k}{k+1}|E|$ edges. $\square$

The main theorem is as follows.

Theorem 4 *Let $G(V, E, F)$ be a geometric plane graph which is 3-edge connected, and let $k \geq 1$ be an integer. There exists a face 6-coloring Λ of G so that the digraph $D'(G; \Lambda)$ is strongly connected, has at most $(1 - \frac{k}{10(k+1)})|E|$ arcs, and its stretch factor is at most $\Phi^2(G)(\Phi(G) - 1)^{2k+4}$.*

Proof Let $G^\star$ be the dual graph of G. Since G is 3-edge connected $G^\star$ is a simple graph and every edge of G is crossed by a unique edge of $G^\star$. Consider an acyclic 5-coloring of $G^\star$ which exists by Theorem 3. Among all ten pairs of colors in the 5-coloring choose a pair so that the forest H induced by vertices colored with these two colors has at least $\frac{|E|}{10}$ edges. By Lemma 4, in this forest we can select a set of induced trees each of diameter at most $4k$ such that they will together span at least $\frac{k}{k+1}\frac{|E|}{10}$ edges of $G^\star$.

We are now ready to color faces of G and define a face 6-coloring Λ of G as follows: We use colors 3 and 4 to color faces corresponding to vertices of trees selected in the dual $G^\star$, and we use colors 1,2,5, and 6 to properly color remaining faces of G.

With $\lambda = 6$, we let color $\alpha = \lceil \frac{\lambda-1}{2} \rceil = 3$ and $\beta = \lceil \frac{\lambda+1}{2} \rceil = 4$. By our construction, the pair $\{\alpha, \beta\}$ must appear at least $\frac{k}{k+1}\frac{|E|}{10}$ times in the face 6-coloring of G. This gives the required bound on the number of arcs of the graph $D'(G; \Lambda)$.

Statements 1 and 2 given in the proof of Theorem 2 imply that if $\{u, v\} \in E$ such that $\Lambda(L_{u,v}) < \Lambda(R_{u,v})$ and if either $\Lambda(L_{u,v}) \neq \alpha$ or $\Lambda(R_{u,v}) \neq \beta$, then $D'(G; \Lambda)$ contains directed path from u to v as well as from v to u. Indeed, if $\Lambda(L_{u,v}) \geq \alpha$, then $\Lambda(R_{u,v}) > \beta$, and we can apply statement 2. Similarly if $\Lambda(R_{u,v}) \leq \beta$, then $\Lambda(L_{u,v}) < \alpha$, and we can apply statement 1. Obviously the pair of colors α and β is not incident to any arc on these paths, so these paths exist in $D'(G; \Lambda)$. Note that these paths have bounded stretch factor as in Theorem 2, in particular $(\Phi(G) - 1)^4$.

To complete the proof we consider $\{u, v\} \in E$ such that $\Lambda(L_{u,v}) < \Lambda(R_{u,v})$ and $\Lambda(L_{u,v}) = \alpha$ and $\Lambda(R_{u,v}) = \beta$. Hence these edges do not occur in $D'(G; \Lambda)$. The edge $\{L_{u,v}, R_{u,v}\}$ of the dual $G^\star$ belongs to one of the selected trees, say T, of diameter at most $4k$. The vertices of this tree correspond to faces of G that are colored with color 3 or 4. Since G is 3-edge connected, there is a path P from u to v in G along these faces such that for each edge of P one of its incident faces has color different from 3 and 4. Hence for each edge of P the digraph $D'(G; \Lambda)$ contains directed paths (in both directions) of length at most $(\Phi(G) - 1)^4$. Since the maximum degree of T is $\Phi(G)$ and the diameter of T is at most $4k$, T has at most $\Phi(G)(\Phi(G) - 1)^{2k}$ vertices. Each of these vertices corresponds to a face of degree at most $\Phi(G)$. Hence the length of P is at most $\Phi^2(G)(\Phi(G) - 1)^{2k}$. Finally we conclude that $D'(G; \Lambda)$ contains a directed path from u to v and from v to u both of length at most $\Phi^2(G)(\Phi(G) - 1)^{2k+4}$.

It follows that $D'(G; \Lambda)$ is strongly connected and has the stretch factor at most $\Phi^2(G)(\Phi(G) - 1)^{2k+4}$. $\square$

Note that with a more careful counting argument the bound on the stretch factor in Theorem 4 can be improved by at least half. Using the following theorem we can further decrease the number of arcs in a strong orientation of G and still keep stretch factor bounded.

Theorem 5 *Let $G = (V, E, F)$ be a 3-connected plane graph. Then G contains a spanning 2-edge connected subgraph G' with at most $|E| - \lfloor \frac{|E|+3}{3\Phi(G)} \rfloor$ edges and $\Phi(G') \leq 2(\Phi(G - 1)$.*

Proof Since G is 3-connected, the dual $G^\star$ is a simple graph. Let T be a spanning tree of $G^\star$. Obviously, the maximum degree of T is at most $\Phi(G)$ and T has least $\frac{|E|}{3}+2$ vertices. The later follows from Euler's formula and the fact that minimum degree of G is 3. Moreover, T has a matching of size at least $\lfloor \frac{|E|+3}{3\Phi(G)} \rfloor$. Indeed, one can obtain such a matching M by recursively adding a pendant edge (an edge adjacent to a leaf) of remaining components of T into M and then removing all remaining edges incident to this edge. Each such operation adds one edge into M and removes at most $\Phi(G)$ edges (including the edge itself) from T. Since T has $\frac{|E|}{3} + 1$ edges at the beginning, the bound follows.

To obtain G', we merge corresponding faces in G for every edge in M. Since M is a matching $\Phi(G') \leq 2(\Phi(G) - 1)$ and G' will be 2-connected. Obviously G' has required number of edges. $\square$

5 Conclusion

We presented algorithms for directing edges of a plane graph having no cut edges such that the resulting digraph is strongly connected and has bounded stretch factor which depends solely on the size of the faces of the original plane graph. An interesting question arises how to construct plane graphs having no cut edges. Although it is well-known how to construct such plane spanners starting from a set of points (e.g., Delaunay triangulation) there are no known constructions in the literature of "local" spanners from UDGs which also guarantee planarity and 2-edge connectivity at the same time.

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