STAT 380: Spring 2018

Midterm Examination # 2: Solutions

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1. Suppose that X and Y are independent random variables. Assume that X has an $\mathrm{Exp}(\lambda_x)$ distribution and Y has an $\mathrm{Exp}(\lambda_y)$ distribution.

   (a) What is $P(Y > X|X = x)$? [3 marks]

   When you condition on $X = x$ you can replace X by x in the event described so:

$$\begin{aligned} P(Y > X|X = x) &= P(Y > x|X = x) \\ &= P(Y > x) \text{ because } Y \text{ and } X \text{ are independent} \\ &= e^{-\lambda_y x}. \end{aligned}$$

   Note that this formula is meaningless for $x < 0$.

   (b) What is $P(Y > X|X)$? [3 marks]

   To compute a conditional expectation like this you do the calculation in part a, get a formula involving x and replace x by X so

$$P(Y > X|X) = e^{-\lambda_y X}.$$

   (c) Write $P(X < Y)$ as a double integral and show that this integral is

$$\frac{\lambda_x}{\lambda_x + \lambda_y}.$$

   [3 marks]

   We simply integrate the joint density of X and Y over the set $\{x < y\}$. The joint density is

$$f(x, y) = \lambda_x e^{-\lambda_x x} \lambda_y e^{-\lambda_y y} 1(x \ge 0) 1(y \ge 0).$$

   So the probability is

$$\int_0^\infty \int_0^\infty \lambda_x e^{-\lambda_x x} \lambda_y e^{-\lambda_y y} 1(x < y)\, dy\, dx = \int_0^\infty \int_x^\infty \lambda_x e^{-\lambda_x x} \lambda_y e^{-\lambda_y y}\, dy\, dx$$

   The inside integral is the answer to part a except for the x density so this is

$$\int_0^\infty \lambda_x e^{-\lambda_x x} e^{-\lambda_y x}\, dx = \lambda_x \int_0^\infty e^{-(\lambda_x + \lambda_y) x}\, dx$$

   Do this integral to get the answer. (Multiply and divide by $\lambda_x + \lambda_y$ and recognize the exponential $\lambda_x + \lambda_y$ density, for instance.)

(d) Now suppose $X_1, \ldots, X_n$ are a sample from the Exponential(λ_x) distribution and $Y_1, \ldots, Y_m$ are a sample from Exponential(λ_y) distribution. Find

$$P\left(\min\{X_1, \ldots, X_n\} < \min\{Y_1, \ldots, Y_m\}\right).$$

[4 marks]

Let $U = \min\{X_1, \ldots, X_n\}$ and $V = \min\{Y_1, \ldots, Y_m\}$. From class you know that U has an exponential $(n\lambda_x)$ distribution and V has an exponential $(m\lambda_y)$ distribution. So

$$P(U < V) = \frac{n\lambda_x}{n\lambda_x + m\lambda_y}.$$

(e) A testing lab compares two types of machine. Type A machines run for an exponentially distributed amount of time with mean 5 hours; Type B machines also have exponentially distributed running times with mean 8 hours. If the lab starts n Type A machines and m Type B machines at the same instant what is the chance that the first two machines to stop running are both Type A machines. [4 marks]

The probability that the first machine to stop is of Type A is the answer to the previous part. At the instant that machine stops you have $n-1$ Type A and m Type B machine still running. By the memoryless property for the exponential distribution the chance that the second one to stop running is Type A, given that the first one is Type A is the answer to part d with n replaced by $n-1$. So the answer is

$$\frac{n\lambda_x}{n\lambda_x + m\lambda_y} \times \frac{(n-1)\lambda_x}{(n-1)\lambda_x + m\lambda_y}.$$

In this case $\lambda_x = \frac{1}{5}$ while $\lambda_y = \frac{1}{8}$. You don't need to plug those in since no great simplification ensues but you get

$$\frac{64n(n-1)}{(8n+5m)(8(n-1)+5m)}.$$

2. In a certain industry accidents occur according to a Poisson process with rate λ equal to 1/3 per day.

(a) There are 30 days in June. What is the chance that there are no accidents after June 20? (No need to get out a calculator; just give me the simplest formula you can.) [4 marks]

There are 10 days after June 20. The number of accidents for a 10 day period has a Poisson distribution with mean 10/3 so the probability is

$$e^{-10/3}.$$

(b) Given that there are 15 accidents in June what is the chance there are no accidents after June 20? (No need to get out a calculator; just give me the simplest formula you can.) [4 marks]

Given that there are 15 accidents in a period they are scattered like uniform order statistics. For a Uniform distribution on $[0, 30]$ the chance of being bigger than 20 is $1/3$ so the probability we want is the probability that a Binomial(15,1/3) is 0 which is

$$\binom{15}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{15} = \frac{2^{15}}{3^{15}}$$

which is pretty small.

3. If T has a uniform distribution on $[0, 1]$ what is the hazard rate of T? [4 marks]

The density of T is

$$f(t) = 1(0 < t < 1)$$

and the cdf is

$$F(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}.$$

The hazard is

$$h(t) = \frac{f(t)}{1 - F(t)} = \begin{cases} 0 & t < 0 \\ \frac{1}{1-t} & 0 \le t \le 1 \\ \text{undefined} & t > 1 \end{cases}.$$