STAT 380: Spring 2018

Midterm Examination # 1 Solutions

Richard Lockhart 2 February 2018

1. A Markov Chain has state space $\{1, 2, 3, 4, 5, 6, 7\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{8} & 0 & \frac{3}{4} & 0 & 0 & & 0 \\ 0 & 0 & 0 & & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Fill in the 3 missing numbers in the matrix. Then identify all the communicating classes, say which states are transient, and give the period of each state. [8 marks]

The transition matrix is

$$\begin{bmatrix} \frac{1}{8} & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

It is clear that 1 leads to 3, then 3 leads to 7 and 7 leads to 1. Thus these three states communicate. State 1 also leads to 6 and 6 leads back to 3 and as before 3 to 1. None of these states leads to any other state (columns 2, 4, and 5 are all 0 in rows 1, 3, 6, and 7). So one closed communicating class is $\{1, 3, 6, 7\}$. Next 2 leads to 4, 4 leads to 5, and 5 leads to 2 so 2, 4, and 5 all communicate. (State 2 also leads to 6 but 6 leads only to itself and to 3.) Thus the two communicating classes are $\{1,3,6,7\}$ and $\{2,4,5\}$. $\{6\}$.

For each of states 1, 3, 6 and 7 it is possible to stay in that state: $P_{ii} > 0$ for $i \in \{1,3,6,7\}$; thus each of these states has period 1. If you don't go to state 6 from any of 2, 4, or 5 then you must have 2 goes to 4, 4 goes to 5 and 5 goes to 2 then the cycle repeats. So the period of these 3 states is 3. I expect this part of the problem to be hard; it is worth 1 mark to get this period correct.

In fact this question was quite well done overall.

2. Players A and B alternate throwing darts at a balloon, trying to break the balloon. Player A starts and has chance p_A of breaking the balloon on that throw. If Player A doesn't break the balloon then Player B throws and has chance p_B of breaking the balloon. They go on like this alternating with the chance of breaking the balloon staying at p_A for player A and p_B for player B every time. What is the chance that player A eventually breaks the balloon? [6 marks]

Let π_A be the probability that A eventually breaks the balloon and π_B be the probability that B breaks it eventually. Then $\pi_B = 1 - \pi_A$ because you are certain to break the balloon eventually unless both $p_A = 0$ and $p_B = 0$ in which case $\pi_A = \pi_B = 0$.

I will use first step analysis. Think about what happens on the first round. The chance that A breaks the balloon on A's first toss is p_A. Given that A does not break the balloon the chance that B breaks the balloon is p_B. The probability that A breaks the balloon eventually is the sum of the probability that A breaks the balloon on the first throw plus the probability that neither A nor B breaks the balloon on the first round and then A eventually breaks the balloon. Given that neither A nor B breaks the balloon on the first round the chance that A breaks the balloon is π_A. So:

$$\pi_A = p_A + (1-p_A)(1-p_B)\pi_A.$$

Rewrite this as

$$\{1-(1-p_A)(1-p_B)\}\,\pi_A = p_A.$$

Now solve and simplify to get

$$\pi_A = \frac{p_A}{p_A + p_B - p_A p_B}.$$

To do this with notation let C be the event that neither player breaks the balloon on the first round. Let C_A be the event that A breaks the balloon on the first round. So

$$P(C_A) = p_A$$

and

$$P(C|C_A^c) = 1 - p_B.$$

So

$$P(C) = (1-p_A)(1-p_B).$$

Let W be the event that A breaks the balloon eventually. Then

$$W = (W \cap C_A) \cup (W \cap C)$$

so, since the two events are disjoint,

$$P(W) = P(W \cap C_A) + P(W \cap C)$$

Then you use

$$W \cap C_A = C_A$$

and

$$P(W \cap C) = P(W|C)P(C).$$

The key point is that

$$P(W|C) = P(W) = \pi_A$$

because if neither player breaks the balloon in the first round you are back where you started.

This gives

$$\pi_A = p_A + (1 - p_A)(1 - p_B)\pi_A$$

as before.

Many people added up a geometric series to do this problem and that is ok but I am not putting the method in these solutions.

3. I have two routes to work. Route A costs me about \$5 to drive and is busy about 2/3 of the time. Route B costs about \$8 to drive and is busy about 1/3 of the time. If I drive a route one day and find it busy I switch to the other route for the next day. Describe a Markov Chain for this model and specify the transition matrix. [4 marks]

   Let X_n denote the route I use on day n. Then X_n is a Markov chain with states A and B. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

4. If I start out driving on route A what is the chance that I drive route B two days later? [4 marks]

   I am asking you for the A, B entry in $\mathbf{P}^2$. I find that

$$\mathbf{P}^2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

   so the answer is 2/3, the upper right corner of this matrix.

5. In the long run how often do I drive route A? [4 marks]

The stationary distribution α satisfies $\alpha = \alpha\mathbf{P}$ and $\alpha_A + \alpha_B = 1$. The equations are

$$\begin{aligned}\alpha_A &= \frac{1}{3}\alpha_A + \frac{1}{3}\alpha_B \\ \alpha_B &= \frac{2}{3}\alpha_A + \frac{2}{3}\alpha_B \\ \alpha_A &= 1 - \alpha_B\end{aligned}$$

The first equation gives $2\alpha_A = \alpha_B = 1 - \alpha_A$ or $3\alpha_A = 1$. So

$$\alpha = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

In the long run I drive route A about 1 day in 3.

Some people noticed the two rows of the transition matrix are the same. They were right to say that this must mean that each row is the stationary initial distribution.

6. In the long run what does it cost me to drive to work per day? [4 marks]

In the long run you spend \$5 on 1 day in 3 and \$8 2 days in 3 so the expectation is

$$\frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 8 = \$7.$$