Name: ____________________ **Student Number**: ____________________

STAT 380: Spring 2016

Final Examination

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Instructions: This is a closed book exam. You are permitted to use 6 sheets of notes, machine-written or hand-written. You may use both sides of the sheets and I place no limits on font size. Calculators are not permitted nor are any other electronic aids. The exam is out of 50. You should have a total of 10 pages; the first page is a grade sheet and the last is extra space. I will be marking for clarity of explanation as well as correctness. Without a clear explanation you should not expect to get more than half marks. You have 3 hours.

1a		3	1b		3	1c		4
2		10				3		10
4a		3	4b		4	4c		3
5a		3	5b		3	5c		4

Total		**50**

Name: ______________ **Student Number**: ______________

1. Each week I buy 0 or 1 or 2 lottery tickets. To decide how many tickets to buy this week I look at how many I bought last week and toss a fair coin. If I bought 0 tickets last week and I get a Heads then I buy 1 ticket next week otherwise I buy 0. If I bought 1 ticket last week and I get a Head I buy 2 tickets this week otherwise I don't buy any. Finally if I bought 2 tickets last week and get a Head I buy two tickets again this week. Otherwise I buy one ticket. Let X_n be the number of tickets I buy in week n. Assume that $X_0 = 0$.

   (a) Write out the transition matrix of the resulting Markov Chain. [3 marks]

   (b) In the long run how many tickets do I buy per week on average? [3 marks]

(c) If each ticket costs \$1 and wins \$10 with probability 1/20 how much money do I lose per week (losses are cost of tickets minus winnings) in the long run? [4 marks]

Name: ________________ **Student Number**: ________________

2. Suppose the $X(t)$ is a continuous time Markov Chain with infinitesimal generator $\mathbf{R}$. Assume that the entries in each column of $\mathbf{R}$ add up to 0. Prove that the transition matrix $\mathbf{P}(t)$ is doubly stochastic for each t; that is, prove that each column of $\mathbf{P}(t)$ adds up to 1. I suggest you use Kolmogorov's Backwards Equations. [10 marks]

Name: ____________ **Student Number**: ____________

3. Particles are detected in a detector according to a Poisson process with rate λ. In one 24 hour period a total of 8 particles are detected. What is the conditional probability that none of these 8 were detected in the last 8 hours of the 24? [10 marks]

Name: ____________________ **Student Number**: ____________________

4. On the midterm I asked you about an experimental design strategy called play-the-winner and asked you to analyze it with a discrete time Markov Chain. I now want you to consider a continuous time version. Two players, A and B, play a game. On each turn of the game one player 'serves' and can either score a point on that turn or not. If the player who served scores a point she serves again. If not, no point is scored and the other player begins to serve. Now suppose that playing each point takes a random amount of time and that this time has an exponential distribution. Suppose that when A serves the time taken for a turn has rate v_A and that she scores a point with probability p_A. Suppose similarly that when B serves she scores a point with probability p_B and that the time taken for a turn has rate v_B.

   This system can be analyzed using a suitable Markov Chain. Use 4 states, say $\{0, 1, 2, 3\}$. In states 0 or 1 player A serves while in states 2 or 3 player B serves. When A serves and wins the state moves to state 1 if the chain is in state 0 and to state 0 if it is in state 1. Similarly when B wins while serving the state changes from 2 to 3 or 3 to 2. When A loses her serve the chain moves to state 2 and when B loses her serve the chain moves to 0.

   (a) What is the matrix $\mathbf{R}$, the infinitesimal generator, for this chain? [3 marks]

Name: ______________ **Student Number**: ______________

(b) Indicate clearly how you would calculate the long run fraction of time during which the players are playing turns where A served. I want to see you convince me you know exactly what equations to solve without solving them. More explicit answers will get more marks. [4 marks]

(c) On the midterm I asked you to find the fraction of turns where A is serving. If v_A is larger than v_B will the answer to part (b) of this question be larger than the answer on the midterm or smaller? Why? [3 marks]

Name: ____________________ **Student Number**: ____________________

5. A continuous time Markov Chain has states $\{0, 1, 2\}$. If the chain is started in chain i it stays there for a time T_i and then moves to a new state with transition matrix $\mathbf{P}$ given by

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

The departure rates from state i are 4, 5, and 6 per time unit for states 0, 1, and 2. That is, the rate for holding time T_0 is 4, for T_1 the rate is 5, and the rate for T_2 is 6.

   (a) Find the instantaneous generator, $\mathbf{R}$, for this chain. [3 marks]

(b) Find the stationary initial distribution for this chain. [3 marks]

(c) Define μ_{ij} to be the mean time for a chain started in state i to reach state j for the first time. Find a formula of the form

$$\mu_{0,2} = a + b\mu_{1,2}$$

and give the values of a and b in terms of the rates and transition probabilities given above. [4 marks]

Name: ____________________ **Student Number:** ____________________

Extra Space