Name: ________________________ Student Number ____________

STAT 380

Final Examination

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Instructions: This is an open book exam. You may use notes, books and a calculator. The exam is out of 50. I will be marking for clarity of explanation as well as correctness.

1	2	3	4	5	6	7	Total

1. A commuter has two possible routes to work, A and B. There is construction activity on route A about 1 day in 20, and on route B about 1 day in 10. If the commuter takes route A and finds construction she switches to route B for the next day otherwise she uses A again. If the commuter takes route B and finds construction she switches to route A for the next day; otherwise she uses B again.

   In the long run on what fraction of days does she commute via route A and on what fraction of her trips does she find construction? Your answer must set down clearly what assumptions you are making to answer the question. Answers with inadequate explanations will get low marks. [9 marks]

Extra space for Q1

2. Each day a random number of newspapers arrive at my house. The probability that k papers arrive is $p_k, k = 0, 1, \ldots$ At the end of the day I may or may not decide to throw out some of the papers that have accumulated. Given that there are m papers the probability that I throw j of them out is $1/(m+1)$ for $j = 0, \ldots, m$. Let X_n be the number of papers left at the end of day n (after I do the throwing out).

   (a) What must you assume in order to have X_n be a Markov chain? [2 marks]

   (b) Suppose U is a random variable with

$$P(U = i) = \frac{1}{m+1}; \qquad i = 0, \ldots, m$$

   Show that $\mathrm{E}(U) = m/2$. [2 marks]

(c) Let μ be the expected number of papers delivered to my house on a given day. Show that [3 marks]

$$\mathrm{E}(X_{n+1}|X_n) = \frac{X_n + \mu}{2}$$

(d) Let m be the long run expected number of papers I have at the end of a day. That is

$$m = \lim_{n\to\infty} \mathrm{E}(X_n)$$

Show that $m = \mu$. [3 marks]

3. Suppose a Markov Chain has state space $\{0, 1, 2\}$ and transition matrix

$$\begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 1/4 & 3/4 \end{bmatrix}$$

and initial distribution $(1/5, 2/5, 2/5)$.

(a) Find $P(X_2 = 0, X_1 = 0, X_0 = 0)$ [1 mark]

(b) Find $P(X_1 = 0 | X_2 = 0, X_0 = 0)$ [2 marks]

(c) Find the stationary initial distribution for this chain. [2 marks]

(d) Starting from state 0 how many steps do you expect to need to get to state 2. [3 marks]

4. Cases of a rare medical condition arrive at a hospital at the rate of 5 per week. Assuming that these arrivals form a Poisson process and that last week there were actually 6 cases what is the conditional probability that all 6 occurred on Saturday or Sunday? [5 marks]

5. A container contains n particles; some are black and the rest are white. Assume that collisions between particles in the container occur at the times of a Poisson process with rate λ. When a collision occurs it is equally likely to be between any of the $n(n-1)/2$ pairs of particles. When two black particles collide there is a chance p that (precisely) one will turn white (instantaneously); otherwise they both stay black. When two white particles collide there is a chance q that (precisely) one will turn black (instantaneously); otherwise they both stay white. When two opposite colour particles collide there is a chance $r/2$ that they both become white and a chance $r/2$ that they both become black.

   (a) Describe an appropriate Markov Chain to analyze this system. Give the transition matrix for the skeleton chain, the instantaneous transition rates and the mean holding times in each state. [5 marks]

(b) For $n = 3$ what fraction of the time are all three balls the same colour? [3 marks]

6. Suppose $X(t); t \geq 0$ is a standard Brownian motion. Fix $0 < a < b$.

(a) Show that [3 marks]

$$P(\max\{X(s)\colon a \leq s \leq b\} > x | X(a) = y)$$
$$= \begin{cases} 1 & y > x \\ 2\,P\left\{N(0,1) > (x-y)/\sqrt{b-a}\right\} & y < x \end{cases}$$
$$= \begin{cases} 1 & y > x \\ 2\Phi\left\{\frac{y-x}{\sqrt{b-a}}\right\} & y < x \end{cases}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du$$

(b) Use the formula in (a) to compute [3 marks]

$$P(\max\{X(s) \colon a \le s \le b\} > x)$$

A correct answer will involve an integral with the function Φ in it; DO NOT try to do the integral.

7. Suppose $N(t); t \geq 0$ is a homogeneous Poisson Process with rate λ. Find a function $w(t)$ such that

$$X(t) = \{N(t) - \lambda t\}^2 - w(t)$$

is a martingale (and prove that the w you have found works). [4 marks]