STAT 380: Spring 2018

Final Examination Solutions

Richard Lockhart 15 April 2018

Notes on the solutions: The most important point is that I have not checked these solutions since I wrote them before giving the exam. In marking I sometimes discover problems with my solutions and fix them on the spot without ever correcting the on-line version. No-one currently knows if that has happened here.

1. Particles pass through a detector at the times of a Poisson Process with a rate of 10 particles per hour. The detector does not detect all the particles; each time a particle goes through there is a 1/5 chance that the particle is detected.

   (a) What is the chance that 3 particles will be detected in the first half hour? [3 marks]

   The number detected forms a Poisson process with rate

   $$\lambda p = 10 \cdot \frac{1}{5} = 2.$$

   The number of points in an interval of length 0.5 hours has a Poisson distribution with parameter

   $$10 \cdot \frac{1}{5} \cdot \frac{1}{2} = 1.$$

   So

   $$P\left(N(0,0.5] = 3\right) = \frac{1^3}{3!}e^{-1} = \frac{e^{-1}}{6}.$$

   (b) Given that 2 particles are detected in the first 30 minutes what is the chance that a total of 6 particles passed through the detector? [3 marks]

   The number of undetected particles is a Poisson Process with rate

   $$\lambda(1-p) = 10\frac{4}{5} = 8$$

   This process is independent of the number detected. If N_d is the number detected and N_u the number undetected then

   $$\begin{aligned} P(N(0,0.5] = 6 | N_d(0,0.5] = 2) &= P(N_u(0,0.5] + N_d(0,0.5] = 6 | N_d(0,0.5] = 2) \\ &= P(N_u(0,0.5] = 4 | N_d(0,0.5] = 2) \\ &= \frac{(8 \cdot 0.5)^4}{4!} e^{8 \cdot 0.5} \\ &= \frac{4^3}{3!} e^{-4} \\ &= \frac{32}{3} e^{-4} \end{aligned}$$

(c) Given that 1 particle is detected in the first 30 minutes what is the probability that a total of 2 particles passed through the detector in the first hour? [3 marks]

Method 1: The total number detected is the sum

$$N = N_d(0, 0.5] + N_u(0, 0.5] + N_d(0.5, 1] + N_u(0.5, 1].$$

So the event

$$N = 2, N_d(0, 0.5] = 1$$

is the same as the event

$$N_u(0, 0.5] + N_d(0.5, 1] + N_u(0.5, 1] = 1, N_d(0, 0.5 = 1]$$

The 4 random variables which add up to N are all independent so

$$\begin{aligned} &P(N(0,1] = 2 | N_d(0, 0.5] = 1) \\ &\qquad = \frac{P(N_u(0, 0.5] + N_d(0.5, 1] + N_u(0.5, 1] = 1, N_d(0, 0.5] = 1)}{P(N_d(0, 0.5] = 1)} \\ &\qquad = P(N_u(0, 0.5] + N_d(0.5, 1] + N_u(0.5, 1] = 1). \end{aligned}$$

All the variables in that sum are independent Poissons so the total has a Poisson distribution. Its mean is (see part b)

$$4 + 1 + 4 = 9$$

so

$$\begin{aligned} P(N(0,1] = 2 | N_d(0, 0.5] = 1) &= P(N_u(0, 0.5] + N_d(0.5, 1] + N_u(0.5, 1] = 1) \\ &= \frac{9^1 e^{-9}}{1!} = 9e^{-9}. \end{aligned}$$

Method 2: we want

$$P(N(0,1] = 2 | N_d(0, 0.5] = 1)$$

The event $\{N(0,1] = 2\} \cap \{N_d(0, 0.5] = 1\}$ is the union of two events:

$$E_1 = \{N(0, 0.5] = 2\} \cap N(0.5, 1] = 0 \cap \{N_d(0, 0.5] = 1\}$$

and

$$E_2 = \{N(0, 0.5] = 1\} \cap \{N(0.5, 1] = 1\} \cap \{N_d(0, 0.5] = 1\}.$$

Rewrite E_1 as

$$\{N_u(0, 0.5] = 1\} \cap N(0.5, 1] = 0 \cap \{N_d(0, 0.5] = 1\}$$

These three pieces are independent so the probability of the intersection is just

$$P(E_1) = P(N_u(0, 0.5] = 1)P(N_d(0, 0.5] = 1)P(N(0.5, 1] = 0)$$

The three variables are Poisson with means 4, 1, and 5 in that order so

$$P(E_1) = \frac{4^1}{1!}e^{-4} \cdot \frac{1^1}{1!}e^{-1} \cdot \frac{5^0}{0!}e^{-5} = 4e^{-10}.$$

Similarly

$$P(E_2) = \frac{4^0}{0!}e^{-4} \cdot \frac{1^1}{1!}e^{-1} \cdot \frac{5^1}{1!}e^{-5} = 5e^{-10}.$$

Thus

$$P(N(0, 1] = 2|N_d(0, 0.5] = 1) = \frac{P(E_1) + P(E_2)}{P(N_d(0, 0.5] = 1)} = \frac{9e^{-10}}{e^{-1}} = 9e^{-9}.$$

2. As in the previous problem particles pass through a detector at the times of a Poisson Process with a rate of 10 particles per hour. This time the detector has three states. When you turn it on it is in the low sensitivity state. In this state each particle is detected with probability 1/5. When the detector is in this low sensitivity state and it detects a particle it changes immediately to a medium sensitivity stat where it detects particles with chance 3/5. In that state it waits till it first detects a particle then switches to a high sensitivity state where no particles go undetected. When a particle is detected in the high sensitivity state it goes immediately back to the low sensitivity state, and so on.

   (a) Define a continuous time Markov Chain for the state of the detector. [3 marks]

   Let $X(t)$ denote the state of the detector with 0 for low sensitivity, 1 for medium and 2 for high. The model says the holding time in state 0 has an exponential distribution with rate

   $$v_0 = \frac{1}{5}10 = 2$$

   The holding time in state 1 is exponential with rate

   $$v_1 = \frac{3}{5}10 = 6$$

   and that for state 2 is

   $$v_2 = 10.$$

   You don't need to give the v values for full marks here – just be clear about what the process itself is – name the 3 states and say what you are tracking.

(b) For the process you have just described specify the elements of the infinitesimal generator $\mathbf{R}$, and the transition matrix of the skeleton chain. [3 marks]

The model says the holding time in state 0 has an exponential distribution with rate

$$v_0 = \frac{1}{5} 10 = 2$$

The holding time in state 1 is exponential with rate

$$v_1 = \frac{3}{5} 10 = 6$$

and that for state 2 is

$$v_2 = 10.$$

The diagonal entries of $\mathbf{R}$ are the negatives of these numbers. The skeleton chain is simple. It moves from 0 to 1 to 2 to 0 to 1 to 2 and so on. So

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Finally for $i \neq j$ we have $q_{ij} = v_i \mathbf{P}_{ij}$ so

$$\mathbf{R} = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -6 & 6 \\ 10 & 0 & -10 \end{bmatrix}$$

(c) In the long run what fraction of the time does the chain spend in each state? [3 marks]

You need to solve the equations

$$\pi \mathbf{R} = 0$$

and

$$\sum_i \pi_i = 1.$$

to find the stationary initial distribution π. The equations are

$$\begin{aligned} 0 &= -2\pi_0 + 10\pi_2 \\ 0 &= 2\pi_0 - 6\pi_1 \\ 0 &= 6\pi_1 - 10\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2 \end{aligned}$$

Thus

$$\pi_2 = \frac{1}{5}\pi_0$$

and

$$\pi_1 = \frac{1}{3}\pi_0$$

so that

$$1 = \pi_0 + \frac{1}{3}\pi_0 + \frac{1}{5}\pi_0 = \frac{23}{15}\pi_0$$

giving

$$\pi_0 = \frac{15}{23} \quad \pi_1 = \frac{5}{23} \quad \pi_2 = \frac{3}{23}.$$

These are the fractions of time spent in each state.

(d) In the long run how many particles are detected per hour? [3 marks]

Method A: in the long run you are in state i a fraction π_i of the time and while you are in state i you detect particles at the rate λp_i where p_i is the detection probability for state i. So the average rate is

$$\lambda(\pi_0 p_0 + \pi_1 p_1 + \pi_2 p_2) = 10\left(\frac{15}{23}\frac{1}{5} + \frac{5}{23}\frac{3}{5} + \frac{3}{23}\frac{5}{5}\right).$$

This simplifies to

$$\frac{90}{23}.$$

Method B: Each time we are in state 0 we spend an exponential amount of time with mean $\frac{1}{2}$ before detecting 1 point. Then we spend an exponential amount of time with mean $\frac{1}{6}$ to detect the next point followed by an exponential amount of time with mean $\frac{1}{10}$ to detect the 3rd point. The total amount of time to detect the first $3n$ points is the sum of n copies of each of these 3 kinds of exponential random variables. So points per time is

$$\frac{3n}{\text{Sum of 3 totals}} = \frac{3}{\text{Sum of 3 averages}}$$

As $n \to \infty$ these averages converge to their expected value so the limit is

$$\frac{3}{\frac{1}{2}+\frac{1}{6}+\frac{1}{10}} = \frac{3}{\frac{15+5+3}{30}} = \frac{90}{23}.$$

(e) In the long run what fraction of the particles are detected? [3 marks]

In the long run you detect particles at the rate 90/23 per hour but they arrive at the rate 10 per hour so you detect

$$\frac{90/23}{10} = \frac{9}{23}$$

of them.

Alternative Method: While in state i you detect particles with probability p_i. You are in state i with probability π_i so the chance you detect a typical particle is

$$\pi_0 p_0 + \pi_1 p_1 + \pi_2 p_2 = \frac{15}{23}\frac{1}{5} + \frac{5}{23}\frac{3}{5} + \frac{3}{23} \cdot 1 = \frac{3+3+3}{23} = \frac{9}{23}.$$

(f) Does the skeleton chain have a stationary distribution? If so what is it? [3 marks]

The equation

$$\pi \mathbf{P} = \pi$$

becomes

$$\begin{aligned} \pi_0 &= \pi_2 \\ \pi_1 &= \pi_0 \\ \pi_2 &= \pi_1 \end{aligned}$$

The three probabilities are all equal, then. Since they sum to one, each must be $1/3$.

(g) Is the skeleton chain aperiodic? If it has a period more than 1, what is it? [3 marks]

It is not aperiodic. The skeleton simply goes 0 to 1 to 2 to 0 to 1 to 2 and so on so the period is 3.

(h) After lunch I come back to discover that the detector is in the medium sensitivity state. How long should I expect to wait until it is next in the low sensitivity state? [3 marks]

By the lack of memory property you wait in this state for an exponential amount of time with rate 6. Then you move to the high sensitivity state and remain there for an exponential amount of time with rate 10. The expected total is

$$\frac{1}{6} + \frac{1}{10} = \frac{8}{30} = \frac{4}{15}.$$

3. In a certain casino there are 3 games, A, B, and C. I start out playing A and win with probability $1/5$ every time I play. I bet a dollar each time and if I win I get the dollar back and 4 more. Every time I win I toss a fair coin. If it comes up heads I move to game B; otherwise I continue to play game A. Game B has a $1/10$ chance of winning. I bet a dollar each time and if I win I get the dollar back and 8 more. Again, when I win I toss a fair coin. If it comes up heads I move to game C; otherwise I continue to play game B. Game C has a $1/20$ chance of winning. I bet a dollar each time and if I win I get the dollar back and 8 more. Again, when I win I toss a fair coin and go back to playing A if I win.

(a) Define a suitable discrete time Markov Chain for this gambling scheme. [3 marks]

The state of the system will be the game I am playing. So X_0 =A and then I move to a new state when I win and toss heads.

(b) What is the transition matrix for the chain. [3 marks]

$$\mathbf{P} = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} & 0 \\ 0 & \frac{19}{20} & \frac{1}{20} \\ 0 & \frac{39}{40} & \frac{1}{40} \end{bmatrix}$$

(c) In the long run, on what fraction of the bets I make am I playing game C? [3 marks]

Find the stationary initial distribution by solving

$$\pi \mathbf{P} = \pi$$

and

$$\pi_A + \pi_B + \pi_C = 1$$

The answer to the question will then be π_C.
The equations are

$$\begin{aligned} \pi_A &= \frac{9}{10}\pi_A + \frac{1}{40}\pi_C \\ \pi_B &= \frac{1}{10}\pi_A + \frac{19}{20}\pi_B \\ \pi_C &= \frac{1}{20}\pi_B + \frac{39}{40}\pi_C \\ 1 &= \pi_A + \pi_B + \pi_C \end{aligned}$$

We see from the first equation that

$$\pi_C = 4\pi_A$$

and from the second that

$$\pi_B = 2\pi_A.$$

So

$$\pi_A + 2\pi_A + 4\pi_A = 1$$

which makes

$$\pi_A = \frac{1}{7} \quad \pi_B = \frac{2}{7} \quad \pi_C = \frac{4}{7}.$$

The answer to the question is that I play game C 4 times in 7 in the long run.

(d) In the long run, how much money do I lose per dollar bet. [3 marks]

My expected loss when playing A is

$$\ell_A = \frac{4}{5} \cdot 1 - \frac{1}{5} \cdot 4 = 0.$$

When playing B it is

$$\ell_B = \frac{9}{10} \cdot 1 - \frac{1}{10} \cdot 8 = \frac{1}{10}.$$

When playing C it is

$$\ell_C = \frac{19}{20} \cdot 1 - \frac{1}{20} \cdot 18 = \frac{1}{20}.$$

In the long run I lose

$$\pi_A \ell_A + \pi_B \ell_B + \pi_C \ell_C = 0 + \frac{2}{7} \cdot \frac{1}{10} + \frac{4}{7} \cdot \frac{1}{20} = \frac{2}{35}$$

dollars per bet.