Robustness and Information Processing

Kenneth Kasa[1]
Department of Economics
Simon Fraser University[2]
e-mail: kkasa@sfu.ca

June, 2003

Abstract. This paper considers a standard Kalman filtering problem subject to model uncertainty and information-processing constraints. It draws a connection between robust filtering (Hansen and Sargent (2002)) and Rational Inattention (Sims(2003)). Considered separately, robustness and Rational Inattention are shown to be observationally equivalent, in the sense that a higher filter gain can either be interpreted as an increased preference for robustness, or an increased ability to process information. However, it is more interesting to consider them jointly. In this case, it is argued that an increased preference for robustness can be interpreted as an increased demand for information processing, while Sims' model of Rational Inattention can be interpreted as placing a constraint on the available supply. This suggests that the way agents actually implement robust decision rules is by allocating some of their scarce information processing capacity to problems that are characterized by high degrees of model uncertainty and risk-sensitivity.

JEL Classification #'s: C61, D81.

[1]I would like to thank Noah Williams for helpful comments on an earlier version of this paper.

[2]Please address correspondence to: Kenneth Kasa, Department of Economics, Simon Fraser University, 8888 University Drive, Burnaby, BC V5A 1S6

Robustness and Information Processing

Kenneth Kasa

1 Introduction

This paper attempts to relate two apparently distinct literatures. One is the so-called 'robust control' literature. Robust control methods were developed by engineers during the 1980s, and are designed to make traditional Linear-Quadratic control more robust to model misspecification. Rather than specify an explicit distribution for a model's disturbances, robust control methods are based on a worst-case analysis of the disturbances, and can be implemented by solving dynamic zero-sum games. Hansen and Sargent (2002) have pioneered the application of these methods in economics. Economists are attracted by robust control because it apparently provides a workable formalization of Knightian Uncertainty.[3]

The second literature is based on the work of Sims (1998, 2003). Sims argues that the dynamic behavior of prices, wages, and macroeconomic aggregates is inconsistent with *both* Classical and Keynesian models of the business cycle. Loosely speaking, prices and wages seem to be too 'sticky' to be consistent with Classical flexible price models. At the same time, quantities seem to be too sticky to be consistent with Keynesian models. Sims goes on to argue that the traditional device of tacking on 'adjustment costs' is ad hoc and unconvincing. As an alternative, he proposes that economists start thinking about the limited ability of agents to process information. Sims shows that we can model these constraints by thinking of agents as finite capacity information transmission channels. Like robust control, this approach has the advantage of drawing on a well developed engineering literature. Although still quite preliminary, the results of Sims suggest that information processing constraints can potentially explain the simultaneous inertia of both prices and quantities.

Although apparently quite distinct, this paper will argue that these two literatures are in fact quite closely related. In particular, I argue that in a sense they are duals of each other, in the same sense that utility and expenditure functions are duals of each other. This duality is based on their alternative interpretations of 'measurement error'. Robust filtering views measurement error as subject to misspecification. Since this misspecification can in principle feedback on the dynamics of the underlying hidden states, it can produce

[3]See Hansen, Sargent, Turmuhambetova, and Williams (2001) and Epstein and Schneider (2001) for a debate about the links between robust control and Knightian Uncertainty.

a quite complicated nonlinear measurement error process, which can be poorly captured by linear least squares projections. To guard against these misspecifications, a robust filter minimizes *maximum* forecast errors, rather than mean-squared forecast errors. This can be implemented by solving a dynamic zero-sum game, in which a malevalent nature (called the 'evil agent' by Hansen and Sargent (2002)) chooses a measurement error sequence to maximize state estimation errors, while at the same time the agent chooses a state estimate to minimize them. The only constraint the evil agent faces is a bound on the relative entropy of the errors.

Sims' (2003) notion of Rational Inattention is based on a quite different interpretation of measurement error. He points out that if we regard individuals as finite capacity information transmission channels (in the sense of Shannon (1948)), then measurement error is *unavoidable* and *rational*. Simply put, measuring a (real-valued) stochastic process without error requires an *infinite* amount of information processing capacity. With a finite capacity, the agent must tolerate measurement errors. The problem he faces is to minimize the sum of squared forecast errors subject to a *lower* bound on the variance of the measurement error. This bound decreases with channel capacity.

Notice the symmetry between these two very different interpretations of measurement error. The evil agent in a robust filter *maximizes* forecast errors subject to an *upper* bound, whereas a person with finite Shannon capacity *minimizes* forecast errors subject to a *lower* bound. Under certain conditions these two problems produce the same observable behavior. These conditions restrict the maximum degree of model uncertainty and the minimum degree of Shannon capacity.

One payoff from making this connection is that it opens the door to an alternative method of parameterizing a preference for robustness. Not surprisingly, the set of potential models an agent insures himself against can have a major influence on his decision rules. Moreover, as noted by Hansen, Sargent, and Tallarini (1999), the parameter which governs the preference for robustness and the degree of model uncertainty can interact with other behavioral parameters in a way that creates potential identification problems. Hence, one would like to be able to restrict this parameter in some way.[4] To date, the only way to do this is by appealing to the 'detection error probabilities' of Anderson, Hansen, and Sargent (2000).

[4]Remember Lucas' admonition (as cited in Sargent (1993)) - 'Beware of theorists bearing free parameters'.

Anderson et. al. argue that agents should only consider potential misspecifications that would have been difficult to detect using observed historical data. This detection error probability decreases as one entertains a larger class of potential models (i.e., as the preference for robustness increases). Essentially, as the class of admissable perturbations increases, it becomes easier to detect them. Hence, given priors about what constitutes a reasonable detection error probability, one can back out a reasonable degree of model uncertainty.

Rather than linking robustness and model uncertainty to detection errors, this paper suggests that we can link them to information processing capacity. The idea that individuals have limits on their ability to process information, and that somehow we can estimate this capacity, goes back at least to the 1950s. Psychologists, in particular, were quick to recognize the potential value to them of Shannon's path-breaking work.[5] Largely due to the efforts of Marschak (1971), economists have also thought about the applicability of Shannon's work. It's probably fair to say, however, that at least so far these efforts have not born fruit.

One problem with estimating information processing capacity that came out of the psychology literature is that this capacity tends to be highly specific to context and experience. That is, we cannot expect to ever arrive at some sort of universal "bits-per-second" number that constrains all individuals in all circumstances.[6] For example, concluding that a given decision rule is consistent with a capacity of 10 bits per time period might be highly dependent on the range of *other* decisions confronting the agent, assuming that capacity is optimally allocated among decisions.[7] Still, even a context specific number could be useful for some questions, as long as this context remains invariant when predicting responses to other changes.

Besides the work of Sims (1998, 2003), two other studies have recently applied the tools of information theory to economics. Moscarini (2002) argues that information processing constraints deliver an improved model of price stickiness. As in this paper, Moscarini works in continuous-time. However, his observer equation implies that finite capacity agents must

[5]See Pierce (1980, chpt. 7) for a review.

[6]For example, the classic paper by Miller (1956) showed that channel capacity varies with the number of underlying alternatives.

[7]Although the absolute scale is arbitrary, information is conventionally defined in base 2 units (called bits), so that the uncertainty of an event or random sequence can be interpreted as the number of true/false questions needed to fully resolve the uncertainty. For example, the result of a single coin flip conveys one bit of information, while knowledge of a single letter conveys $\log_2 26 = 4.7$ bits. See, e.g., Cover and Thomas (1991).

sample *discretely*. Agents with higher capacity can sample more frequently. He points out that this approach might be less susceptible to the Lucas Critique than existing models of price stickiness, since channel capacity is more likely to remain invariant to policy changes. Turmuhambetova (2003) develops a model of *endogenous* capacity constraints. She introduces an "information processing cost" into the agent's objective function, and allows the agent to augment capacity subject to a (utility) cost. This makes capacity state- and time-dependent, and can be interpreted as a formalization of Kahneman's (1973) notion of 'elastic capacity'.

The remainder of the paper is organized as follows. The next section outlines Sims' model of Rational Inattention, and provides some necessary background on information theory. Section 3 develops and solves a standard Kalman filtering problem under alternative assumptions about model uncertainty and information processing. Section 4 briefly discusses how the duality between robustness and attention can be used to parameterize a preference for robustness. Caveats and possible extension are discussed in the Conclusion, and an Appendix contains proofs of some technical results.

2 Information Processing and Rational Inattention

2.1 Background

The word "seminal" is often used to describe influential scientific work. In few instances is it as apt as in the case of Shannon's (1948) work on information theory. Shannon's innovation was to view information as a *stochastic process*, which enabled him to quantify the amount of information in terms of the probability distribution generating the data. Loosely speaking, the greater the range of potential outcomes of an experiment or measurement, the greater is the information conveyed by the results. This idea led him to define the key concepts of an **information transmission channel** and **channel capacity**. According to Shannon, a transmission channel can be viewed abstractly as any mapping between inputs (e.g., analog voice data in a telephone receiver) and observed or measured outputs (e.g., what you hear at the other end). From a statistical perspective, a channel is just a conditional probability distribution. Since most real world channels are corrupted by noise, Shannon went on to define the capacity of a channel to be the maximum rate at which signals can be transmitted through the channel with arbitrarily small detection error. In the case of Gaussian channels, where both the signal and the (additive) noise are normally distributed, capacity is proportional to the difference between the log of the unconditional variance of the signal and the log of its variance conditional on the set of observed channel outputs. Hence, it is closely related to the familiar concept of a signal-to-noise ratio.[8]

Besides its revolutionary impact on its intended field of application (i.e., telecommunications engineering), Shannon's work has also deeply influenced such diverse fields as physics, psychology, statistics, and linguistics. Wiener (1961) was the first to recognize its potential use to social scientists. When combined with his own work on signal processing, Wiener thought it would create an entirely new field of study, which he called "cybernetics".

Despite being the offspring of such illustrious parents as Shannon and Wiener, cybernetics has never really taken off in the social sciences. As it turns out, human beings are far more complicated than telephones! Psychologists were the first to encounter the difficulties in applying Shannon's work to humans. Not surprisingly, debate arose very early about how to map Shannon's concepts of transmission channels and capacity into human decision-makers.

Broadbent (1958) took the natural first step. He assumed humans are exactly like Shannon's telephone receivers. According to Broadbent, information processing ability is governed by a single all-encompassing channel, which processes incoming data in a serial manner. This approach quickly ran into empirical problems, however, since experiments often revealed that capacity is not immutable. For example, practice and experience often increase the measured capacity to perform various stimulus-response tasks.

Moray (1967) maintained the hypothesis of a single governing channel, but attempted to account for the variability of measured capacity. He focused on the allocation problem that arises when multiple tasks compete for access to a channel. He argued that since learning leads to more efficient methods of performing a task, it "frees up" capacity for other tasks, and therefore increases overall channel capacity.

The single channel model of human information processing is attractive because it is consistent with the findings of a multitude of task interference experiments. Performance of one task apparently interferes with the simultaneous performance of other tasks. Allport, Antonis, and Reynolds (1972) presented experimental evidence which suggested that this

[8]If C is channel capacity, $H(x)$ is the entropy rate of a source signal, and $H(x|y)$ is the conditional entropy rate of the signal given measured output, y, then by definition, $C = \max[H(x) - H(x|y)]$, where the maximum is taken over all (admissable) source signals. The above statement then follows from the facts that: (1) Gaussian processes maximize entropy subject to power constraints, and (2) Up to an additive constant, the entropy rate of a Gaussian process is the logarithm of its (instantaneous) variance. See Cover and Thomas (1991, chpts. 10 and 11).

doesn't necessarily imply the existence of a single channel. They instead argue that human information processing is more accurately described as the parallel use of several specialized channels. Their experiments revealed strong interference only when stimuli were presented in similar ways. For example, recognition memory is seriously disrupted when stimuli are presented solely in an auditory manner. However, when one stimulus is presented visually and one presented auditorily, interference is greatly reduced (and hence, measured capacity is increased). This led them to posit the existence of parallel channels, corresponding to various physical systems, e.g., auditory, visual, etc.

Kahneman (1973) tried to unite these two models of human information processing. His approach is perhaps most familiar to economists, since he regarded capacity as a scarce resource that must be allocated among competing demands. This is consistent with Moray's earlier work, but makes the allocation process much more explicit. Kahneman was motivated by the experimental results of Posner and Bois (1971), which suggested that people have discretion over the allocation of information processing capacity. Kahneman argued that capacity is "elastic" in the sense that it can expand in response to incentives. His model was the first to explain a very simple but robust experimental finding, i.e., task performance is seldom perfect, even for very easy tasks. According to Kahneman, this is explained by the fact that in most experiments it simply isn't worth it to perform the task perfectly. Kahneman's work implies that the presence of errors does *not* imply that all available capacity is being utilized. This makes the measurement of capacity much more difficult.[9]

Thus far we have been discussing the "macroeconomics" of human information processing. During the past two decades attention has shifted to the "micro-foundations" of information processing. No doubt this switch was partially motivated by diminishing returns, but it was also driven by breakthroughs in neurobiology and medical technology. The hope is that these developments will eventually allow researchers to relate information processing to identifiable and measurable brain activity. Unfortunately for those still interested in more macroscopic questions, this research is still at too early a stage to be readily applied. However, the current gap between the older "reduced form" approach and the newer neurobiological approach has produced a new field called "cognitive neuroscience", which is attempting to bridge the gap. (See Parasuraman (1998) for a wide ranging survey). For the purposes of this paper, however, these more recent literatures can be disregarded, since Sims (2003) appeals to the

[9]As noted in the Introduction, the recent work of Turmuhambetova (2003) can be interpreted as a formalization of these ideas.

older reduced form approach.

2.2 Sims' Model

Motivated by the observed inertia of both prices and quantities, Sims (2003) incorporates an information-processing constraint into a standard Linear-Quadratic Regulator. Like Broadbent and Moray, he assumes the agent has a single channel with a fixed capacity, expressed in bits per time period. Although Sims provides some analysis of the multivariate case and the implied capacity allocation problem, here I present a version of the simple univariate Permanent Income model developed in section 6 of his paper. In this model, the agent only has to decide how much to save each period. Keeping track of his wealth requires some information processing effort.

Viewed as a general LQR problem, the agent wants to control fluctuations in both a state variable and a control variable. However, it is assumed that some information processing effort must be expended to monitor changes in the state variable. With a capacity constraint, there is a maximum effort that can be devoted to observing the state variable. Hence, "measurement errors" emerge endogenously, with a variance that is inversely related to the channel capacity. These measurement errors then confront the agent with a signal extraction problem, which produces exactly the kind of damped, delayed, and smoothed response to shocks that is so commonly seen in the data. Economists usually resort to ad hoc adjustment cost functions to explain this inertia. (See Sims (1998)).

To be specific, assume the agent wants to solve the following problem:

$$\min_{\{u_t\}} E_0 \sum_{j=0}^{\infty} \beta^j (Rx_{t+j}^2 + Qu_{t+j}^2) \tag{1}$$

where x_t is a state variable and u_t is the control variable. The state evolves according to the transition equation:

$$x_t = Ax_{t-1} + Bu_{t-1} + \epsilon_t \tag{2}$$

where $\epsilon_t \sim N(0, \omega^2)$. At this point an exogenously specified 'measurement equation' is typically added to the model. Sims's model of Rational Inattention dispenses with the measurement equation, and instead adds an information-processing constraint.

The Linear-Quadratic structure here continues to deliver Certainty-Equivalence, so the optimal feedback policy is:

$$u_t = -F\hat{x}_t \tag{3}$$

where F is determined by discounted versions of the usual formulas:

$$\begin{aligned} F &= \beta \frac{ABP}{Q + \beta B^2 P} \\ P &= R + \beta P A^2 Q (Q + \beta B^2 P)^{-1} \end{aligned} \tag{4}$$

The novelty of the Rational Inattention model, therefore, is in the construction of the state estimate, $\hat{x}_t$.

Sims proceeds in the following two steps. First, the minimum conditional variance of the state that is consistent with the capacity constraint is calculated. From this, we can then use Kalman filtering along with the state transition equation to infer a minimum implicit measurement error variance.

To solve the first step, remember that based on the work of Shannon we can define the flow of information each period to be the reduction in the entropy of the x_t process. Since ϵ_t is Gaussian, this entropy differential turns out to be just the difference in (log) conditional variances. If σ_t^2 represents the beginning of period conditional variance of x_t, then in the absence of any information processing, the conditional variance of x_{t+1} is:

$$\mathrm{var}_t(x_{t+1}) = A^2\sigma_t^2 + \omega^2 \tag{5}$$

The agent can reduce this conditional variance by expending some monitoring effort. This effort is limited, however, by the constraint that the flow of information, as defined by the reduction in log conditional variances, must not exceed the channel capacity. Letting κ be the channel capacity (in bits per period), we have:[10]

$$\frac{1}{2}\left[\log(A^2\sigma_t^2 + \omega^2) - \log(\sigma_{t+1}^2)\right] \leq \kappa \tag{6}$$

Setting $\sigma_t^2 = \sigma_{t+1}^2$, this constraint implies the following bound on the steady state variance of the state, denoted by $\bar{\sigma}^2$:

$$\bar{\sigma}^2 \geq \frac{\omega^2}{e^{2\kappa} - A^2} \tag{7}$$

Note that if $|A| > 1$ then channel capacity must be at least $\log(A)$, otherwise the variance of the state explodes. Violation of this bound can be interpreted as being a situation where

[10]The scale factor, 1/2, is not really essential. It comes from the 1/2 term in the Gaussian density.

stabilization of the system requires more attention and effort than the agent is capable of.[11] Thus, as in Robust Control, which imposes a lower bound on θ (sometimes called the 'breakdown point'), there are limits to how far we can push the agent away from the standard frictionless benchmark. Likewise, note that as capacity becomes unbounded ($\kappa \rightarrow \infty$), the constraint becomes vacuous. In principle, an infinite capacity agent could eliminate state uncertainty entirely, although this would not be optimal unless $Q = 0$.

The second step is to now use Kalman filtering formulas to infer a minimum measurement error variance. That is, we view the agent as observing $y_t = x_t + \varepsilon_t$, where ε_t is an implicit i.i.d. measurement error. The agent can get a more precise estimate by expending some information processing capacity. However, given the bound, κ, he cannot reduce the conditional variance of the state below the right-hand side of (7). If we let $\bar{\Sigma}$ be the steady state variance of the state estimate, then Kalman filtering delivers the following implicit equation for $\bar{\Sigma}$:

$$\bar{\Sigma} = [A^2\bar{\Sigma} + \omega^2]\left[\frac{\mathrm{var}(\varepsilon)}{A^2\bar{\Sigma} + \omega^2 + \mathrm{var}(\varepsilon)}\right] \tag{8}$$

Solving for $\mathrm{var}(\varepsilon)$ and setting $\bar{\Sigma} = \bar{\sigma}^2$ yields the following bound on the variance of the implicit measurement error:

$$\mathrm{var}(\varepsilon) \geq \frac{\bar{\sigma}^2(A^2\bar{\sigma}^2 + \omega^2)}{(A^2 - 1)\bar{\sigma}^2 + \omega^2} \tag{9}$$

where $\bar{\sigma}^2$ is given by the right-hand side of (7). It can be verified from (7) and (9) that as information processing capacity shrinks, the agent must tolerate greater measurement error.

3 Filtering

To highlight the relationship between robustness and attention, this section studies an even more stripped down version of Sims' model, which abstracts from control and just focuses on the filtering half of the problem. In particular, a standard Kalman filtering problem is analyzed under alternative assumptions about model uncertainty and information processing. Later I discuss how control can be re-introduced.

The analysis proceeds in four steps. First, as a benchmark, I lay out the standard continuous-time Kalman-Bucy filtering problem. The next step then incorporates model uncertainty. Following Anderson, Hansen, and Sargent (2003), model uncertainty is represented by a set of candidate models, parameterized by their relative entropies vis-a-vis an

[11] Of course, if $|A| < 1$ the bound is irrelevant, since then the system is stable by itself.

exogenous reference model. The key result here is to show that the filtering algorithm's gain parameter is increasing in model uncertainty. The third section restores confidence in the model, but then introduces a capacity constraint on information processing. A preliminary result first shows that channel capacity is directly proportional to the variance of the state estimate. It is this result that makes continuous-time such a convenient assumption. As a direct corollary, it is then easy to show that the filtering gain increases with channel capacity. Taken together, the results of sections 3.2 and 3.3 point to an observational equivalence between a preference for robustness and an increased capacity to process information. This suggests some potentially intriguing psychological links between uncertainty and information processing. The fourth part explores these connections by incorporating both model uncertainty and information processing constraints. It is argued that we can interpret a desire for robustness as giving rise to a demand for information processing, which is constrained by the available channel capacity.

3.1 The Kalman-Bucy Filter

Consider the following pair of stochastic differential equations:

$$dX = -aXdt + \sigma dW^1 \tag{10}$$

$$dY = Xdt + dW^2 \tag{11}$$

where $W^1(t)$ and $W^2(t)$ are independent standard Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$, on which is defined the filtration $\mathcal{F}_t = \sigma(X_0, W^1(s), W^2(s); 0 \leq s \leq t)$, where $X_0 : \Omega \rightarrow R^1$ is a Gaussian random variable representing the unknown initial value of X, which is assumed to be independent of W^1 and W^2. $\{\mathcal{F}_t\}$ is assumed to satisfy the 'usual conditions', e.g., right-continuity and augmentation by P-negligible sets. All processes are assumed to be $\mathcal{F}_t$-adapted.

Equation (10) defines the law of motion for an unobserved state variable, $X(t)$, and equation (11) defines the measurement, or observer, equation. In what follows, let $\mathcal{Y}_t = \sigma(Y(s); 0 \leq s \leq t)$ be the filtration generated by the history of the $Y(t)$ process.

Usually it wouldn't matter if we wrote the measurement equation in a slightly different way, relating X to the *level* of Y as opposed to its 'derivative',

$$Y(t) = X(t) + W^2(t) \tag{12}$$

In the context of information processing, however, there is a substantive difference between these two specifications. Writing the observer equation as in (12) would imply an *infinite* rate of information transimission. In continuous-time, the signal must be 'smoother' than the noise (see, e.g., Sims (1998)). One way to respect this limitation is to assume the agent samples the process at discrete intervals. This is the route taken by Moscarini (2002). In this paper, I pursue an alternative approach. I write the observer equation as in (11) and assume the current signal influences the instantaneous rate of change in the measured output. This ensures the signal is smoother than the noise, which becomes clear when we interpret the observer equation as an integral equation (which is the mathematically preferred interpretation):

$$Y(t) = Y(0) + \int_0^t X(s)ds + \sigma \int_0^t dW_s^2 \tag{13}$$

The agent's filtering problem can now be expressed as the following minimum norm problem:

$$\inf_{\hat{X}_t \in \mathcal{X}_t} \|X_t - \hat{X}_t\|^2 = \inf_{\hat{X}_t \in \mathcal{X}_t} \int_\Omega (X_t - \hat{X}_t)^2 dP \tag{14}$$

where the set of admissable estimators is given by:

$$\mathcal{X}_t := \{\hat{X}_t : \Omega \to R^1; \hat{X}_t \in L^2(P) \text{ and } \hat{X}_t \text{ is } \mathcal{Y}_t\text{-measurable}\}$$

The solution of this problem is well known (see Liptser and Shiryaev (2000a) for a derivation). It can be written as the following recursion:

$$d\hat{X} = -a\hat{X}dt + K(t)[dY - \hat{X}dt] \tag{15}$$

with initial condition given by the prior mean of X_0, and where the 'Kalman gain' satisfies the Riccati equation:

$$\dot{K} = -2aK - K^2 + \sigma^2 \tag{16}$$

with initial condition given by the prior variance of X_0. Equations (15) and (16) comprise the famous Kalman-Bucy filter. Notice that as $t \to \infty$ $K(t)$ converges to the following unique positive solution of the algebraic Riccati equation, $0 = -2aK - K^2 + \sigma^2$,

$$\bar{K} = -a + \sqrt{a^2 + \sigma^2} \tag{17}$$

As you would expect, the agent revises his forecasts more in response to new information when the signal-to-noise ratio, σ^2, increases.

3.2 Robust Filtering

The previous analysis was based on the assumption that the model was correctly specified, and the agent knew this. What if this isn't the case? What if the agent entertains doubts about the model? The presumed linear law of motion for the unobserved state variable, $X(t)$, might not only be a poor approximation, but the nature of the misspecification could well be difficult to diagnose. This wouldn't be a serious problem if Kalman filters were robust to such misspecification. Unfortunately, this isn't the case. (See Petersen and Savkin (1999) for examples).

In response to this lack of robustness, engineers have devised control and filtering methods that are more robust to misspecification. The breakthrough came in the control context, with the work of Zames (1981), who showed how to derive policies that *guarantee* a given performance level. His idea was to switch objective functions, from the H^2 sum-of-squares norm to the H^∞ supremum norm. Rather than optimize the average behavior of a system based on a presumed statistical model for the disturbances, H^∞-control and filtering is based on a deterministic, worst-case analysis of the disturbances. This permits a model's error term to capture a complex amalgam of potential misspecifications, e.g., omitted variables, neglected nonlinearities, or misspecified statistical distributions.

Recently, Hansen and Sargent (2002) have begun to import and adapt these methods to economics. Although the formal decision theory justifying their use in economics remains the subject of some dispute (see, e.g., the debate between Epstein and Schneider (2001) and Hansen, Sargent, Turmuhambetova, and Williams (2001)), the hope is that robust control offers a convenient dynamic extension of Gilboa and Schmeidler's (1989) axiomatization of Knightian Uncertainty. This axiomatization is motivated by the desire to explain experimental findings that apparently violate Savage's axioms, especially the Ellsberg Paradox. Gilboa and Schmeidler show that the Ellsberg Paradox can be resolved if the 'Sure Thing Principle' is relaxed.[12]

One aspect of the engineering approach to robust control and filtering that economists have had to modify is its deterministic interpretation of a model's disturbance process. Deterministic errors produce stochastic singularities and make it difficult to compare models. Engineers don't need to worry about this since the appropriate degree of robustness is usually dictated by externally imposed performance or stress requirements. In economics, however,

[12]The Sure Thing Principle is the analog of the independence axiom in models of subjective probability.

agents presumably get to decide the appropriate degree of robustness based on observations of historical data. The greater the range of potential models that are consistent with the data, the greater the degree of model uncertainty.

A stochastic approach to model uncertainty has been developed by Anderson, Hansen, and Sargent (2003).[13] In this approach model uncertainty is introduced by thinking of the agent as being unsure about the true probability measure generating the data. Equations (10) and (11) now constitute only a benchmark, or reference, probability measure around which a class of perturbed probability measures are contemplated. The distance between the reference model and the alternatives is measured by their relative entropies:

Definition 1: *Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathcal{M}(\Omega)$ be a set of probability measures on $(\Omega, \mathcal{F})$. Given any two probability measures $Q, P \in \mathcal{M}(\Omega)$, the relative entropy of the probability measure Q with respect to the probability measure P is defined by*:

$$h(Q \| P) := \begin{cases} \int \log\left(\frac{dQ}{dP}\right) dQ & if \;\; Q \ll P \\ +\infty & otherwise \end{cases}$$

where $Q \ll P$ denotes absolute continuity of the measure Q with respect to the measure P, and $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P.

Hence, relative entropy is akin to a log-likelihood ratio statistic. The absolute continuity condition implies that Q and P share the same measure zero events. This is a reasonable condition to impose on the set of admissable perturbations. If they weren't absolutely continuous, they would be easy to distinguish.

A key advantage of measuring model uncertainty by relative entropy is that Girsanov's Theorem delivers a very convenient parameterization of the alternative models. Let $(\xi_i(t), \mathcal{F}_t)$, $0 \leq t \leq T$ be a random process satisfying the following (Novikov) conditions:

$$\begin{aligned} P\left(\int_0^T |\xi_i(s)|^2 ds < \infty\right) &= 1 \\ E^P \exp\left(\frac{1}{2}\int_0^T |\xi_i(s)|^2 ds\right) &< \infty \end{aligned}$$

From this process, construct the following process,

$$\zeta_i(t) = \exp\left(\int_0^t \xi_i(s) dW_s - \frac{1}{2}\int_0^t |\xi_i(s)|^2 ds\right)$$

[13] Interestingly, engineers are starting to adopt a stochastic approach as well. In fact, the robust filtering approach of Petersen and Ugrinovskii (2002) is actually closer to my approach than is Anderson et. al. (2003).

It can be shown that $\zeta_i(t)$ is a positive, continuous martingale, with $E^P(\zeta_i(t)) = 1\,\forall t$ (see, e.g., Liptser and Shirayev (2000a, ch. 6)), and hence, from Girsanov's Theorem, we have the following relationship between the reference probability measure P and a perturbed probability measure, Q_i, as parameterized by the process $\xi_i(t)$,

$$dQ_i(\omega) = \zeta_i(\omega)dP(\omega)$$

Notice that if $\xi_i(t) = 0\,\forall t$ then $\zeta_i(t) = 1$ and $Q_i = P$. More generally, the process $\xi_i(t)$ parameterizes the distance between Q_i and P. In fact, we have

$$h(Q_i\|P) = E^{Q_i}\int_0^T |\xi_i(s)|^2 ds$$

Applying this change of measure to both the state transition equation in (10) and the measurement equation in (11) produces the following set of drift distorted models:

$$dX = -(aX - \sigma\xi_1(t))dt + \sigma d\tilde{W}^1 \tag{18}$$

$$dY = (X + \xi_2(t))dt + d\tilde{W}^2 \tag{19}$$

where $\tilde{W}^1(t)$ and $\tilde{W}^2(t)$ are Wiener processes on $(\Omega, \mathcal{F}, Q)$, with $Q = Q^1 \times Q^2$, which are related to the reference processes as follows:

$$\begin{aligned}\tilde{W}^1(t) &= W^1(t) - \int_0^t \xi_1(s)ds\\ \tilde{W}^2(t) &= W^2(t) - \int_0^t \xi_2(s)ds\end{aligned}$$

Now, to design a robust filter, the agent employs the device of an 'evil agent', who attempts to subvert the agent's filtering efforts by choosing the worst model for any given filter. In doing this, the only constraint the evil agent faces is a bound on the relative entropy between the reference model and the distorted worst-case model. That is, the agent makes no *a priori* assumptions about the parametric form of potential model misspecification. To allow for an infinite horizon without discounting, I express the relative entropy bound as the following limiting time average:

$$h_\infty(Q\|P) \equiv \limsup_{T\to\infty} \frac{1}{T} E^Q \int_0^T (|\xi_1(s)|^2 + |\xi_2(s)|^2)ds < \mathcal{U} \tag{20}$$

As $\mathcal{U}$ increases the agent is effectively allowing for more model uncertainty. Conversely, as $\mathcal{U} \downarrow 0$, the agent becomes fully confident in the specification of the reference model.

A robust filter can now be characterized as the Nash equilibrium of the following dynamic zero-sum game,[14]

$$V_t = \inf_{\{\hat{X}_s\}} \sup_{Q} \left\{ \limsup_{T\to\infty} \frac{1}{2T} E^Q \int_0^T |X_s - \hat{X}_s|^2 ds - \frac{1}{2}\theta h_\infty(Q\|P) \right\} \tag{21}$$

subject to the distorted model in (18) and (19). The parameter θ can be interpreted as a Lagrange Multiplier on the relative entropy constraint in (20). (As noted by Hansen and Sargent (2002), we can omit the $\theta\mathcal{U}$ term since it does not influence behavior). Hence, θ indexes the degree of robustness. As θ increases, robustness decreases. In the limit, as $\theta \to \infty$, the problem becomes identical to the standard Kalman-Bucy filtering problem studied in the previous section. Conversely, the smallest value of θ that is consistent with the existence of a bounded solution to (21) can be interpreted as a stochastic H^∞ filter, providing the maximal degree of robustness for a given reference model. As emphasized by Anderson, Hansen, and Sargent (2003) this may or may not be an empirically plausible filter. If the implied detection error probability is regarded as being implausibly low, then θ should be increased.

Before computing the equilibrium of this game there is an important feature of (21) that should be highlighted. Notice that the lower limit of integration in (21) is 0, not t, implying the agent cares about *past* forecast errors. Whether this is reasonable is likely to be case specific. As noted by Hansen and Sargent (2002), assuming agents care about past forecast errors is reasonable when agents must commit in advance to a given filter. On the other hand, if agents ignore past forecast errors, letting bygones be bygones, then it turns out that the standard Kalman filter is robust, in the sense that it minimizes the maximum one-step ahead forecast error.[15] In the next section we'll see that a nonrecursive robust filter can be interpreted as requiring extra information processing. This suggests that when agents must commit in advance to a filter they will allocate more information processing to the task.

When solving the robust filtering problem in (21) it proves convenient to exploit the following Legendre-type duality relationship between relative entropy and a risk-sensitive version our criterion function:

[14] Hansen and Sargent (2002) call this the 'multiplier game', while Basar and Bernhard (1995) call it the 'soft-constrained game'.

[15] See Hansen and Sargent (2002) or Basar and Bernhard (1995, ch. 7) for a proof. This result is perhaps most easily understood in the frequency domain. A recursive robust filter minimizes the maximum of the spectral density of one-step ahead forecast errors. However, by construction, the Kalman filter produces a white noise forecast error, which is flat already, and hence, maximally robust.

Lemma 3.1: *Let $P \in \mathcal{M}(\Omega)$, and $\psi : \Omega \to R^1$ be a measurable function. Then for every $Q \in \mathcal{M}(\Omega)$ we have the following Legendre transform relationships between $h(Q\|P)$ and the log moment generating function of ψ:*

$$
\begin{aligned}
h(Q\|P) &= \sup_{\psi \in \mathcal{B}(\Omega)} \left\{ \int \psi dQ - \log\left(\int e^{\psi} dP \right) \right\} && (22) \\
\log\left(\int e^{\psi} dP \right) &= \sup_{Q \in \mathcal{M}(\Omega)} \left\{ \int \psi dQ - h(Q\|P) \right\} && (23)
\end{aligned}
$$

where $\mathcal{B}(\Omega)$ denotes the set of bounded $\mathcal{F}$-measurable functions on Ω.

This duality relationship is useful because it allows us to transform our entropy constrained robust filtering problem into an equivalent risk-sensitive control problem. In particular, we shall use the version in (23) with $\psi \equiv (X_s - \hat{X}_s)^2$ to replace the 'inner part' of the minmax filtering problem with a conventional risk-sensitive objective function. Although the ψ function in our case is quadratic, and hence unbounded, Dai Pra, Meneghini, and Runggaldier (1996) show that the duality in lemma 3.1 can be extended to cover this case as well. Solving the resulting risk-sensitive filtering problem yields,

Proposition 3.1: *If $\theta > 1$ there is a unique solution of the the robust filtering problem in (21) given by*

$$
\begin{aligned}
d\hat{X} &= -a\hat{X}dt + K_r(t)[dY - \hat{X}dt] && (24) \\
\dot{K}_r &= -2aK_r - \left(1 - \frac{1}{\theta}\right) K_r^2 + \sigma^2 && (25)
\end{aligned}
$$

where the robust Kalman gain, $K_r(t)$, converges to the following solution of the algebraic Riccati equation, $\dot{K}_r = 0$:

$$
\bar{K}_r = \frac{-a + \sqrt{a^2 + (1 - \theta^{-1})\sigma^2}}{1 - \theta^{-1}} \tag{26}
$$

If $\theta < 1$ there does not exist a solution of the robust filtering problem.

Proof: To prove this we can follow Ugrinovskii and Petersen (2002). The basic idea is to scale both sides of (23) by θ, and then define $\tilde{\psi} = \theta\psi$. The right-hand side becomes an entropy constrained filtering problem in terms of the scaled objective function, $\tilde{\psi}$. The left-hand side becomes a risk-sensitive control problem, also in terms of $\tilde{\psi}$, with risk-sensitivity parameter $1/\theta$. The solution of this problem is a special case of Theorem 3 in Pan and Basar (1996). □

There are a couple of interesting things to note about the robust Kalman gain in (26). First, it is identical to the robust filter in Theorem 7.6 of Basar and Bernhard (1995), which is based on a deterministic interpretation of the disturbances. Hence, adopting a stochastic approach does not produce different results, but it does aid in the calibration of the θ parameter. Second, comparing (26) to the standard Kalman gain in (17) reveals, as expected, that $K_r \to K$ when $\theta \to \infty$. More generally, we have:

Corollary 3.1: *The gain of the robust filter increases with model uncertainty.*

Proof: Differentiate (26) with respect to θ and verify $\partial K_r/\partial\theta < 0$. □

Since this is the main result from the perspective of relating robustness and information processing, it is useful to have some intuition for it. To do this, suppose for simplicity that the state transition equation is the only source of uncertainty, so that $\xi_2(t) = 0\, \forall t$. Also, without loss of generality (see, Hansen and Sargent (2002)), confine attention to a Markov Perfect Nash equilibrium in which the filterer selects a feedback rule (ie., a Kalman gain parameter, K) at the same time his evil agent selects a feedback rule for $\xi_1(t) = Ge(t)$, where $e(t)$ is the forecast error $(X - \hat{X})$. Given choices of K and G we get the following Ornstein-Uhlenbeck process for the forecast errors:

$$de = -(a - \sigma G + K)edt + \sigma d\tilde{W}_1 - KdW_2$$

which produces the following expression for the steady state variance of the forecast error:

$$\mathrm{var}(e) = \frac{\sigma^2 + K^2}{2(a - \sigma G + K)}$$

Subtracting the relative entropy constraint, $\theta G^2 \mathrm{var}(e)$, and then differentiating with respect to K and G yields the following two reaction functions:

$$K(G): \qquad 2K(a - \sigma G + K) = \sigma^2 + K^2 \tag{27}$$

$$G(K): \qquad \sigma G = (a + K) - \sqrt{(a + K)^2 - \sigma^2/\theta} \tag{28}$$

These two functions are plotted in Figure 1.[16] Starting at the standard Kalman gain, the evil agent can produce a higher forecast error variance by choosing a disturbance process that feeds back positively on e. This increases the persistence and variance of the forecast errors. To ward off this possibility, the agent picks a higher, more vigilant, gain parameter. This

[16]Note, figure 1 just depicts the essential aspects of the situation. $K(G)$ is actually concave (when K is on the horizontal axis) and $G(K)$ is convex.

makes him less susceptible to low frequency misspecifications, which are especially damaging. The agent raises the gain parameter up until the point that an evil agent would no longer have an incentive to increase the persistence of the forecast errors. Clearly, making the evil agent's actions more costly by increasing θ causes the $G(K)$ reaction function to shift down and to the left (as well as becoming flatter). This allows the agent to relax a bit and reduce the filter gain.

A natural question at this point is whether there is any evidence of heightened sensitivity to prediction errors in the presence of model uncertainty. Unfortunately, while there is a vast psychology literature debating whether agents update according to Bayes Rule, it does not directly examine the effects of model uncertainty. The early literature actually found evidence of *conservatism* (Edwards (1968)). In what is perhaps the only analysis of the implications of model uncertainty for Bayesian updating, Navon (1978) argued that apparent conservatism could be explained if agents suspect the data are subject to measurement error or mean reversion. While this would indeed rationalize conservative updating, it would *not* be robust. An agent with a preference for robustness should be concerned that the data are *less* mean reverting than it appears. Failure to respond to hidden persistence is far more costly than failure to detect mean reversion. As it turns out, however, the recent literature has focused much more on the tendency of agents to *overreact* to new information in a variety of settings (e.g., Kahneman and Tversky (1973) and Grether (1980)). This should probably not be construed as evidence in favor of robust filtering, however, since this same evidence points to more radical departures from Bayesian updating. At this point all one can safely conclude is that the jury is still out.[17]

3.3 Capacity Constrained Filtering

Let's now restore the agent's faith in the model. Instead of being unsure about the model, assume he has limits on his ability to process information. The objective is to formalize and generalize the analysis of information processing that was presented earlier in the context of Sims' (2003) model of Rational Inattention. To do this we now regard the state space model in (10) and (11) as an information transmission channel, with output Y and input signal X. Our first task is to provide a rigorous definition of the information about X that

[17] Interestingly, Mullainathan (2002) shows that memory limitations can also produce overreaction to forecast errors. This occurs when new information triggers memories that convey similar information.

is contained in Y.

Definition 2: *Let Y and X be two random processes on $[0,T]$, with joint probability measure $\mu_{X,Y}$ and marginal probability measures μ_Y and μ_X. If $\mu_{X,Y} \ll \mu_X \times \mu_Y$ then the* **mutual information**, $\mathcal{I}_T(X,Y)$, *between Y and X is*:

$$\mathcal{I}_T(X,Y) = \int \ln\left(\frac{d\mu_{X,Y}}{d[\mu_X \times \mu_Y]}\right) d\mu_{X,Y}$$

Referring back to Definition 1, we can see that mutual information is just the relative entropy between the joint density and the product of the marginals. Clearly, if Y and X are independent, so that $d\mu_{X,Y} = d[\mu_X \times \mu_Y]$, then $\mathcal{I}_T(X,Y) = 0$.

Having defined mutual information, we can now define in general terms the capacity of an information transmission channel between Y and X:

Definition 3: *Let Y and X be two random processes on $[0,T]$, with mutual information, $\mathcal{I}_T(X,Y)$. Then the* **capacity**, $\mathcal{C}$, *of a channel between X and Y is*:

$$\mathcal{C} := \sup_{X \in \Lambda} \mathcal{I}_T(X,Y)$$

where Λ defines a set of admissable X processes.

In most applications, the admissable set Λ is defined by power or amplitude constraints. A classic result in information theory, first established by Shannon (1948), is that if Y is Gaussian and Λ is defined by a power constraint then X is also Gaussian. More generally, however, calculating capacity is a challenging calculus of variations problem. Fortunately, for our purposes this is not an issue, since I regard X as exogenous, i.e., outside the agent's control. With this assumption, capacity and mutual information become synonomous.

Our task now is to apply Definition 2 to the particular channel defined by equations (10) and (11). This gives us:

Proposition 3.2: *The mutual information (and capacity) of the transmission channel defined by equations (10) and (11) is given by*:

$$\mathcal{I}_T(X,Y) = \frac{1}{2} E \int_0^T (X_s - \hat{X}_s)^2 ds \tag{29}$$

where $\hat{X}_s = E(X_s | Y_u, 0 \leq u \leq s)$ and is given by the Kalman-Bucy filter in equation (15).

Proof: For our special case of linear diffusion processes, this result was first proved by Duncan (1970). However, I follow the proof in Liptser and Shiryaev (2000b, ch. 16). The details are in the appendix. □

As you would expect from standard signal extraction logic, the information in Y about X is an increasing function of the conditional variance of X. For example, if the conditional variance of X is small then movements in Y likely reflect measurement error, and do not convey much information about X. We can use this result along with equation (17) to derive the following expression for the asymptotic *rate* of information flow (per unit time) in terms of the steady state Kalman gain:

Corollary 3.2: *The steady state rate of information conveyed by Y about X is given by*

$$\lim_{T\to\infty} \frac{1}{T}\mathcal{I}_T(X,Y) = \frac{1}{2}\bar{K} = \frac{1}{2}(-a + \sqrt{a^2+\sigma^2})$$

Proof: $K(T)$ converges to $\bar{K}$ as $T \to \infty$. □

If we now denote channel capacity by κ, the information processing constraint simply becomes $\bar{K} < 2\kappa$. There is certainly no guarantee that this be binding. If the parameters are such that the optimal Kalman gain satisfies this constraint then information processing constraints are irrelevant. Since for our purposes this is not an interesting case, in what follows I make the following assumption about the parameters:

Assumption 1: *The parameter values satisfy the inequality,* $\kappa < \frac{1}{2}(-a + \sqrt{a^2+\sigma^2})$

Hence, decisions characterized by a low signal-to-noise ratio are assumed to have a low channel capacity. Not only is this a reasonable assumption, but it is likely to be the equilibrium outcome in any problem permitting the endogenous allocation of capacity across multiple decisions (eg., Turmuhambetova (2003)).

If Assumption 1 holds then we get the following characterization of capacity constrained filtering.

Proposition 3.3: *If Assumption 1 holds then the filter gain is* 2κ, *and an increase in channel capacity increases the filter gain.*

Proof: If Assumption 1 holds then the capacity constraint binds. Since the loss function is globally convex with a minimum at $(-a+\sqrt{a^2+\sigma^2})$, if the constraint is binding then it is optimal to increase the filter gain in response to an increased channel capacity. □

Relating Proposition 3.3 back to Corollary 3.1 produces the following observational equivalence result, which is the main result of this section.

Corollary 3.3: *If Assumption 1 holds then Rational Inattention and Robust Filtering are observationally equivalent, in the sense that a higher filter gain can either be interpreted as an increased preference for robustness or an increased ability to process information.*

3.4 Capacity Constrained Robust Filtering

The previous two sections examined robust filtering and capacity constrained filtering separately. What if model uncertainty and information processing constraints are present simultaneously? The observational equivalence result in Corollary 3.3 points to a potentially intriguing connection between these two concepts. In particular, since robustness and channel capacity both increase the gain of the filter, the previous results suggest that one way agents might actually *implement* a robust filter is by re-allocating some of their scarce information processing capacity to decisions that demand a relatively high degree of robustness, either because they have a relatively high degree of model uncertainty associated with them, or because they are especially risk-sensitive.

Figure 2 illustrates the essential aspects of the situation. The robust filter gain in equation (26) can be interpreted as giving rise to a 'demand' for information processing as a function of θ. Letting $\kappa^d(\theta)$ denote this function, we have

$$\kappa^d(\theta) = \frac{-a+\sqrt{a^2+\sigma^2(1-\theta^{-1})}}{2(1-\theta^{-1})} \tag{30}$$

As θ decreases the filter gain increases, and along with it the implied mutual information rate and information processing demand. Conversely, as $\theta \to \infty$ the filter converges to the standard Kalman filter, with an implied information rate equal to $\frac{1}{2}\bar{K}$.

Clearly, if Assumption 1 holds, the agent is unable to achieve *any* robustness. He's already at his limit. Hence, Figure 2 depicts a situation where channel capacity exceeds the nonrobust demand (i.e., $\kappa > \frac{1}{2}(-a+\sqrt{a^2+\sigma^2})$). When this is the case the agent can enhance the robustness of the filter, but only up to the point where $\kappa^d(\theta)$ intersects the constraint. Hence, channel capacity limits the achievable robustness of the filter.

4 Using Channel Capacity to Parameterize a Preference for Robustness

One common criticism of robust control and filtering is that it imputes an excessive degree of pessimism to agents. Why base decisions on the worst potential outcome? Didn't Savage show that even when events are unique and nonreplicable we can still model agents as if they were formulating subjective probabilities over them and maximizing expected utility?

To a Bayesian this is the end of the story.[18] To make sense of robust control and filtering you must doubt the validity of the Savage axioms. Cause for doubt has come on two fronts. First, a growing experimental literature has repeatedly confirmed and generalized the results of the Ellsberg paradox, which is fundamentally in conflict with the existence of a (unique) subjective prior probability distribution. Second, and perhaps more persuasive to those who doubt the generality of experimental results, models which incorporate a preference for robustness seem better able to explain observed market data, particularly asset market data.[19]

Of course, a Bayesian response to these results is: How could they do worse? Robust control and filtering models come with an extra free parameter (i.e., θ) that can be tuned to explain anything. Hence, to make robust control methods persuasive it is essential that some discipline be placed on the calibration of θ. Setting θ too low can indeed produce an excessive degree of pessimism.

Fortunately, Anderson, Hansen, and Sargent (2003) have developed a useful strategy for calibrating this parameter. Their strategy exploits a connection between the worst-case shocks and conditional relative entropy on the one hand, and a connection between conditional relative entropy and Bayesian detection error probabilities on the other hand. Given a prior about what constitutes a reasonable detection error probability, they show how to back out a value for θ that generates this probability. With this methodology, one can set θ so that agents only hedge against models that could have plausibly generated past observed data. This disciplines the choice of θ, and answers the criticism that a minimax objective makes people unduly pessimistic.

[18] See Sims (2001) for a critique of robust control along standard Bayesian lines. He argues that while a minimax approach might provide a convenient method for generating a useful prior in some applications, it should not be regarded as a normative decision theory.

[19] See, e.g., Hansen, Sargent, and Tallarini (1999) and Anderson, Hansen, and Sargent (2003), and the references they provide.

Interestingly, information processing constraints suggest an alternative, complementary strategy for calibrating θ. If we actually knew what channel capacity was then equation (30) would deliver an immediate parameterization of θ, as long as we assumed that the capacity constraint was binding. Unfortunately, as noted earlier, it is not likely that we could ever expect to measure directly and persuasively a generally applicable value of κ. This hope was pretty much dead by the early 1960s.[20] However, I will now argue that we can exploit the insights of Anderson, Hansen, and Sargent (2003) to devise an *indirect* strategy for linking θ to channel capacity.

The calibration consists of two steps. The first step is to relate channel capacity to detection error probabilities. This can be done using the results of Kailath (1969) and Evans (1974). The second step is to then use equation (30) to link κ to θ. Hence, as in Anderson, Hansen, and Sargent (2003), we can calibrate θ to detection error probabilities, but we do so through the intermediate step of relating each to channel capacity. As long as you are willing to assume the capacity constraint binds, this indirect strategy can provide some computational advantages.

To see this, let's start with Kailath's (1969) expression for the likelihood ratio between the following two diffusion processes:

$$\begin{aligned} H_0 : dY &= X_{0t}dt + dW_t \\ H_1 : dY &= X_{1t}dt + dW_t \end{aligned}$$

where our task is to decide which of the two unobserved diffusion processes, X_{0t} or X_{1t}, is generating the observed Y data. The likelihood ratio can be written as follows:

$$LR \equiv \Lambda(T) = \exp\left\{\int_0^T (\hat{X}_{1t} - \hat{X}_{0t})dY - \frac{1}{2}\int_0^T (\hat{X}_{1t}^2 - \hat{X}_{0t}^2)dt\right\} \tag{31}$$

where $\hat{X}_{it}$ is a $\mathcal{Y}_t$-measurable least squares estimate of X_{it} assuming that hypothesis H_i is true. These too are diffusions, generated by the Kalman-Bucy filter derived in section 3.1. Given this, the decision rule is standard:

$$\begin{aligned} \ell(T) \equiv \ln\Lambda(T) &< \gamma \qquad \Rightarrow H_1 \\ &> \gamma \qquad \Rightarrow H_0 \end{aligned}$$

[20] Despite this, a still active subfield in psychology (i.e., psychophysics) continues to provide measures of channel capacity in very specific, narrowly defined contexts.

where γ is a threshold determined by the relative importance of Type I and Type II errors. In what follows, I treat the two errors symmetrically and set $\gamma = 0$.

In principle, we could at this point derive a diffusion process for $\ell(T)$ and calculate the probabilities that $\ell(T) < 0$ when H_0 is true and $\ell(T) > 0$ when H_1 is true. This would give us the overall detection error probability. Unfortunately, exact expressions are hard to come by. In practice, it is much easier to calculate *bounds* on the detection errors. Essentially, we need to calculate the probability that a diffusion with positive drift ends up at a negative value. This is a 'large deviations' problem, with the escape probability being an exponentially decreasing function of the sample size and a 'rate function' determined by the parameters of the problem. This is basically the route taken by Evans (1974), although he works with the Fokker-Planck equation for the moment generating function of $\ell(T)$ rather than its Legendre transform.

Specifically, if we define the moment-generating function of $\ell(T)$ as follows,

$$M_i(s) = E\{\exp[s\ell(T)|H_i]\} = E[\Lambda(T)^s|H_i]$$

then a standard Chernoff bound calcuation yields,

$$P(\text{error}|H_i) \leq M_i(s)$$

The trick is to evaluate $M_i(s)$. By first writing the prediction error decompositions of Y under each hypothesis, and then substituting into (31), we get:

$$\Lambda_t^s \equiv \phi(t) = \exp\left[s\int(\hat{X}_{1t} - \hat{X}_{0t})dv - \frac{s}{2}\int(\hat{X}_{1t} - \hat{X}_{0t})^2 dt\right]$$

where dv is a Brownian motion prediction error process. Applying Ito's lemma to this yields:

$$d\phi = \frac{s^2 - s}{2}(\hat{X}_{1t} - \hat{X}_{0t})^2\phi dt + s(\hat{X}_{1t} - \hat{X}_{0t})\phi dv \tag{32}$$

Using (32) along with the hypothesized diffusions for X_1 and X_2 delivers

$$M_T(s) = \exp\left[\frac{s^2 - s}{2}\int_0^T P(t)dt\right]$$

where $P(t)$ is the solution of a Riccati equation. Finally, noting that $s = 1/2$ is the bound-minimizing choice of s, and letting $T \to \infty$ gives us the following detection error probability:

$$P(\text{error}) \leq \exp\left[-\frac{1}{8}P_\infty T\right] \tag{33}$$

where P_∞ is the steady state value of $P(t)$.

Now the punch-line is that by following the logic of section 3.3 we can regard P_∞ as the capacity of a re-defined information transmission channel. As this capacity increases the maximal detection error probability decreases. This suggests an *upper* bound on P_∞ which, from (30), implies a *lower* bound on θ (assuming $\kappa^d(\theta) < P_\infty$). Notice that we have not had to calculate relative entropy as a function θ. Although it's lurking in the background as an element of P_∞, as long as we assume that channel capacity is binding we do not have to explicitly calculate it. This may be advantageous in some applications.

5 Conclusion

This paper has illustrated some potential connections between Robust Filtering and Rational Inattention. It has shown that under certain conditions a greater responsiveness to new information can either be interpreted as an increased concern for robustness in the presence of model uncertainty, or an increase in information processing ability when agents are regarded as finite capacity information transmission channels. By making these connections, this paper raises questions about the deeper psychological links between uncertainty and information processing. Following up on the early work of Kahneman (1973), perhaps one way agents actually formulate and implement robust policies is by reallocating their limited channel capacity to highly uncertain or risk-sensitive situations.

Besides exploring these links in greater detail, there are a number of other more straightforward extensions that could be pursued in future work. First, while it is usually the case that there is little conceptual loss of generality in confining attention to univariate cases, that is not true in models of Rational Inattention. As noted by Sims, and discussed in more detail by Kahneman, with Rational Inattention there may be possibilities to reallocate capacity among variables, and focus attention where it has the greatest payoff. This calls for a multivariate extension of these results. Second, this paper has focused on filtering and abstracted from control. Information processing constraints also have implications for control that merit future research. As noted by Hansen and Sargent (2002) and Basar and Bernhard (1995), with model uncertainty the usual separation between filtering and control no longer applies, and it is likely that information processing constraints also produce some interesting interactions between these two problems. In addition to the work of Sims, interesting starts along this dimension have been made by Turmuhambetova (2003) and Kuznetsov, Liptser, and Serebrovskii (1980). Finally, the observational equivalence derived here bears some resemblance to the result of Hansen, Sargent, and Tallarini (1999). They showed in the context of a Permanent Income model that the quantity implications of robustness are observationally equivalent to a reduced rate of time preference. They argue that the effects of robustness and model uncertainty manifest themselves most clearly in asset prices. This suggests that information processing constraints might also have important implications for security market data and the resolution of various asset price anomalies.

APPENDIX

This appendix fills in some of the details of the proof of Proposition 3.2. First, one can readily verify that equations (10) and (11) satisfy conditions (A) - (E) in chapter 7 of Liptser and Shiryaev (2000a).[21] Given this, the fact that $\mu_{X,Y} \ll \mu_X \times \mu_W$ and $\mu_Y \ll \mu_W$ implies

$$\frac{d\mu_{X,Y}}{d[\mu_X \times \mu_Y]} = \frac{d\mu_{X,Y}/d[\mu_X \times \mu_W]}{d\mu_Y/d\mu_W} \tag{A1}$$

Next, we can apply lemmas 7.6 and 7.7 in Lipster and Shiryaev (2000a) to conclude,

$$\frac{d\mu_{X,Y}}{d[\mu_X \times \mu_W]} = \exp\left[\int_0^T X_t dY_t - \frac{1}{2}\int_0^T X_t^2 dt\right] \tag{A2}$$

$$\frac{d\mu_Y}{d\mu_W} = \exp\left[\int_0^T \hat{X}_t dY_t - \frac{1}{2}\int_0^T \hat{X}_t^2 dt\right] \tag{A3}$$

Substituting (A2) and (A3) into (A1) yields,

$$\begin{aligned} \ln \frac{d\mu_{X,Y}}{d[\mu_X \times \mu_Y]} &= \int_0^T [X_t - \hat{X}_t] dY_t - \frac{1}{2}\int_0^T [X_t^2 - \hat{X}_t^2] dt \\ &= \int_0^T \left([X_t - \hat{X}_t] X_t - \frac{1}{2}[X_t^2 - \hat{X}_t^2] \right) dt + \int_0^T [X_t - \hat{X}_t] dW_t \end{aligned} \tag{A4}$$

Finally, taking expectations in (A4), collecting terms, and exploiting the properties of stochastic integrals to eliminate the last term in (A4) yields,

$$E\left[\ln \frac{d\mu_{X,Y}}{d[\mu_X \times \mu_Y]}\right] = \frac{1}{2} E \int_0^T (X_t - \hat{X}_t)^2 dt$$

[21] These conditions include the following: (A), existence of a strong (i.e., $\mathcal{F}_t^{X,W}$-measurable) solution of (11); (B), Nonanticipative drift and diffusion coefficients in (11); (C), A growth condition on the diffusion coefficient in (11) (satisfied trivially here); and boundedness assumptions on the drift term in (11). Actually, since W^2 and X are assumed independent, we can (from the note to Theorem 7.23 in Lipster and Shiryaev) dispense with conditions (A) and (C).

Figure 1

Nash Feedback Reaction Functions

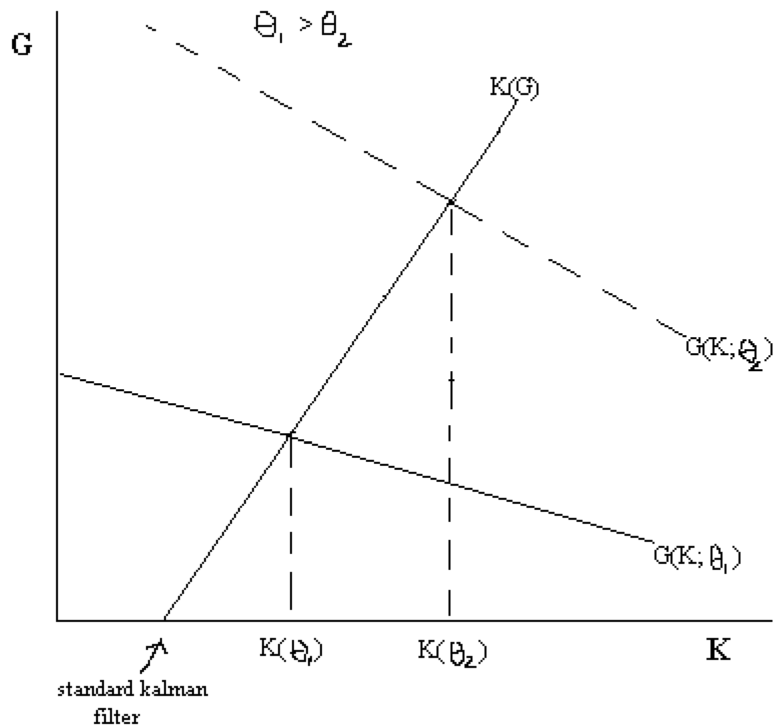
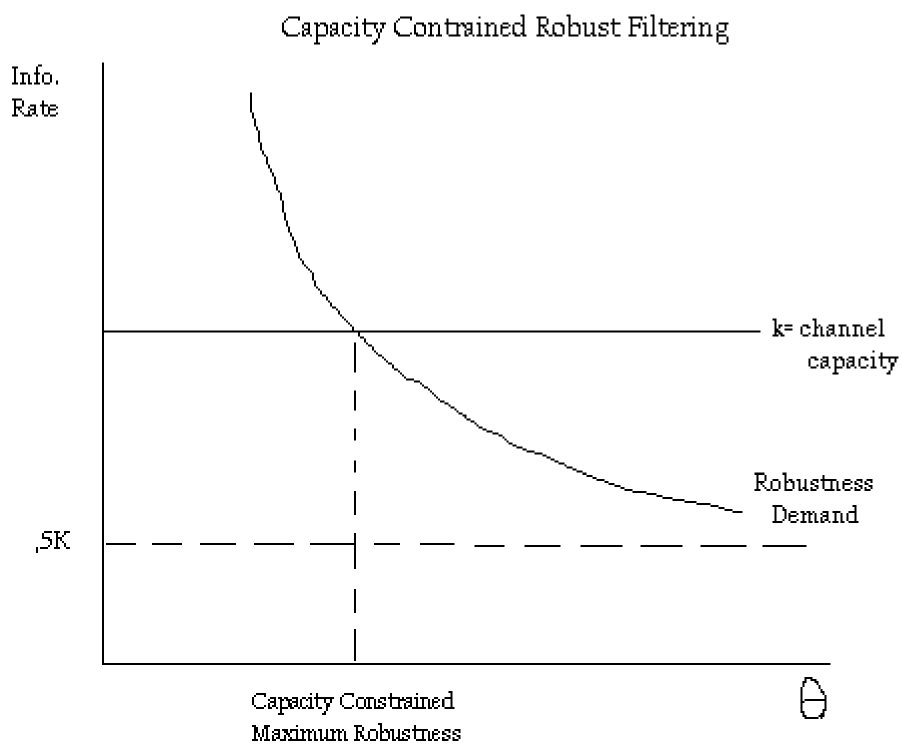


Figure 2


Capacity Contrained Robust Filtering
Info.
Rate
k= channel
capacity
Robustness
Demand
.5K
Capacity Constrained
Maximum Robustness
θ

REFERENCES

Allport, D.A., B. Antonis, and P. Reynolds, 1972, On the Division of Attention: A Disproof of the Single Channel Hypothesis, *Quarterly Journal of Experimental Psychology* 24, 225-35.

Anderson, Evan W., Lars P. Hansen, and Thomas J. Sargent, 2003, A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection, mimeo, University of Chicago.

Basar, Tamer, and Pierre Bernhard, 1995, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd Edition, Boston-Basel-Berlin: Birkhauser.

Broadbent, Donald E., 1958, *Perception and Communication*, London: Pergamon.

Cover, Thomas M., and Joy A. Thomas, 1991, *Elements of Information Theory*, John Wiley & Sons, Inc.

Dai Pra, P., L. Meneghini, and W. Runggaldier, 1996, Connections Between Stochastic Control and Dynamic Games, *Mathematics of Control, Systems, and Signals* 9, 303-26.

Duncan, Tyrone E., 1970, On the Calculation of Mutual Information, *SIAM Journal of Applied Mathematics* 19, 215-20.

Edwards, Ward, 1968, Conservatism in Human Information Processing, in B. Kleinmuntz, ed., *Formal Representation of Human Judgment*, Wiley & Sons.

Epstein, Larry G., and Martin Schneider, 2001, Recursive Multiple-Priors, mimeo, University of Rochester.

Evans, James E., 1974, Chernoff Bounds on the Error Probability for the Detection of Non-Gaussian Signals, *IEEE Transactions on Information Theory* 20, 569-77.

Gilboa, Itzhak, and David Schmeidler, 1989, Maximin Expected Utility with Non-unique Prior, *Journal of Mathematical Economics* 18, 141-53.

Grether, David M., 1980, Bayes' Rule as a Descriptive Model: The Representativeness Heuristic, *Quarterly Journal of Economics* 95, 537-57.

Hansen, Lars P., and Thomas J. Sargent, 2002, *Robust Control and Economic Model Uncertainty*, forthcoming monograph.

Hansen, Lars P., Thomas J. Sargent, and Thomas D. Tallarini, 1999, Robust Permanent Income and Pricing, *Review of Economic Studies* 66, 873-907.

Hansen, Lars P., Thomas J. Sargent, Gauhar A. Turmuhambetova, and Noah Williams, 2000, Robustness and Uncertainty Aversion, mimeo, University of Chicago.

Kahneman, Daniel, 1973, *Attention and Effort*, Prentice-Hall, Inc.

Kahneman, Daniel, and Amos Tversky, 1973, On the Psychology of Prediction, *Psychological Review* 80, 237-51.

Kailath, Thomas, 1969, A General Likelihood-Ratio Formula for Random Signals in Gaussian Noise, *IEEE Transactions on Information Theory* 15, 350-61.

Kuznetsov, N.A., R.S. Liptser, and A.P. Serebrovskii, 1980, Optimal Control and Data Processing in Continuous Time, *Avtomatika i Telemekhanika* 10, 47-53.

Liptser, Robert S., and A.N. Shiryaev, 2000a, *Statistics of Random Processes I: General Theory*, Second Edition, Springer-Verlag.

____________, 2000b, *Statistics of Random Processes II: Applications*, Second Edition, Springer-Verlag.

Marschak, Jacob, 1971, Economics of Information Systems, *Journal of the American Statistical Association* 66, 192-219.

Miller, George A., 1956, The Magic Number Seven, Plus or Minus Two: Some Limits on our Capacity for Processing Information, *Psychological Review* 63, 81-97.

Moray, N., 1967, Where is Capacity Limited? A Survey and a Model, *Acta Psychologica* 27, 84-92.

Moscarini, Giuseppe, 2002, Limited Information Capacity as a Source of Inertia, mimeo, Yale University.

Mullainathan, Sendhil, 2002, A Memory-Based Model of Bounded Rationality, *Quarterly Journal of Economics* 117, 735-66.

Navon, David, 1978, The Importance of Being Conservative: Some Reflections on Human Bayesian Behavior, *British Journal of Mathematical and Statistical Psychology* 31, 33-48.

Pan, Zigang, and Tamer Basar, 1996, Model Simplification and Optimal Control of Stochastic Singularly Perturbed Systems Under Exponentiated Quadratic Cost, *SIAM Journal on Control and Optimization* 34, 1734-66.

Parasuraman, Raja, 1998, *The Attentive Brain*, MIT Press.

Petersen, Ian R., and A.V. Savkin, 1999, *Robust Kalman Filtering for Signals and Systems with Large Uncertainties*, Birkhauser: Boston.

Petersen, Ian R., and V.A. Ugrinovskii, 2002, Robust Filtering of Stochastic Uncertain Systems on an Infinite Time Horizon, *International Journal of Control* 75, 614-26.

Pierce, John R., 1980, *An Introduction to Information Theory: Symbols, Signals and Noise*, Second Edition, Dover.

Posner, Michael I., and S.J. Boies, 1971, Components of Attention, *Psychological Review* 78, 391-408.

Sargent, Thomas J., 1993, *Bounded Rationality in Macroeconomics*, Oxford Univ. Press.

Shannon, Claude E., 1948, A Mathematical Theory of Communication, *Bell System Technical Journal* 27, 379-423, 623-656.

Sims, Christopher A., 1998, Stickiness, *Carnegie-Rochester Conference Series on Public Policy* 49, 317-56.

____________, 2001, Pitfalls of a Minimax Approach to Model Uncertainty, *American Economic Review* 91, 51-54.

____________, 2003, Implications of Rational Inattention, *Journal of Monetary Economics*

Turmuhambetova, Gauhar A., 2003, Stochastic Control for Systems with Information Constraints, mimeo, University of Chicago.

Wiener, Norbert, 1961, *Cybernetics: or Control and Communication in the Animal and the Machine*, 2nd Edition, MIT Press.

Zames, George, 1981, Feedback and Optimal Sensitivity: Model Reference Transformation, Multiplicative Seminorms, and Approximate Inverses, *IEEE Transactions on Automatic Control* 26, 301-20.