

THE FIRM'S PROFIT MAXIMIZATION PROBLEM

These notes are intended to help you understand the firm's problem of maximizing profits given the available technology. Both a general algebraic derivation of the problem and the optimality conditions and specific numerical examples are presented. This is done separately for the short and long run.

Profit Maximization - Overview

We assume that firms are in business to make as much money as possible, i.e. they strive to maximize their profits. This assumption has its rationale in the idea of "natural selection" or "survival of the fittest" - if a firm is not maximizing profits its competitors who do would eventually drive it out of business by employing more efficient (and more profitable) methods of production. Even if the firm is monopoly, however it would most probably want to maximize profits - after all firms are owned by people from whom we assumed more is always better. Thus more firm's profit would mean more income (or wealth) for its owners which makes them happier. Thus profit maximization seems a reasonable assumption about firms' behavior. The firm maximizes profits (revenues minus costs) by choosing the most efficient way to produce, i.e. by choosing the optimal amounts of the factors of production to employ. The firm chooses these optimal amounts taking into account the available technology embodied in the production function which gives the relationship between the amounts of inputs put into production and the maximum possible amount of output that can be produced.

Short vs Long Run

The firm's problem of maximizing profits differs between the short and the long run. Remember that we call "the short run" a time period in which at least some factors of production are fixed. Thus in the short run the firm is unable to vary all factors - some of them are fixed at predetermined levels and cannot be changed. Thus the only thing that the firm can choose in this case is the quantities of **variable inputs** (not fixed) to hire. For example we can imagine that we produce output using labor and a plant and the size of the plant is fixed in the long run. The only choice variable for the owner of such firm in the short run (the only thing that can be varied to maximize profits) is then the number of people hired.

In contrast, in the long run all factors are variable so the firm can choose how much to hire of **all inputs**, which makes its problem a bit more complicated. In our example above, the owner of the firm can choose both the number of people to hire and the size (or number) of plants.

Notation

To simplify things, in the discussion below we will use an example with 2 inputs and one output. We will use the following notation:

- x_1, x_2 - the quantities of the two inputs hired by the firm

- $y = f(x_1, x_2)$ – the amount of output produced from x_1 and x_2 where $f(x_1, x_2)$ is the production function
- w_1, w_2 - the per unit market prices (wages) for the two inputs
- p - the market price of output

We will assume that the firm takes input and output prices w_1, w_2 and p as given - i.e. it cannot influence them. Thus we are looking at the case where both input and output markets are **competitive** (each firm is too small to affect the prices).

Profit Maximization in the Short Run

Since we are in the short run (SR) assume that factor 2 for example is fixed, i.e. $x_2 = \bar{x}_2$ (we just have our single factory). The firm's problem then is to maximize profits by choice of x_1 - the amount of input 1 to be hired. Notice that the firm cannot choose anything else here - prices are assumed to be taken as fixed for the firm, \bar{x}_2 is fixed and output is just a function of x_1 and \bar{x}_2 i.e. once we choose x_1 we have chosen y as well.

The firm's problem is then:

$$\max_{x_1} pf(x_1, \bar{x}_2) - w_1x_1 - w_2\bar{x}_2$$

In words, the above says - maximize profits (=revenue (py) minus costs ($w_1x_1 + w_2\bar{x}_2$)) by choosing the optimal quantity of input 1 (x_1) to hire. It looks like the consumer problem but there is no constraint. Why? Aren't firms constrained? Yes, they are, the constraint is actually there - it is just plugged in into the profit function - remember that the constraint on firm's behavior is put by nature - it is the production function, saying that output, y must equal $f(x_1, \bar{x}_2)$ - i.e. that y cannot be chosen independently of x_1 .

How do we solve it?

This is a simple maximization problem of one variable so we just need to take the first derivative of the function that is being maximized and set it equal to zero. We get:

$$pf_1(x_1, \bar{x}_2) - w_1 = 0 \tag{1}$$

where we denote by $f_1(x_1, \bar{x}_2)$ the derivative of the production function with respect to x_1 . Notice that \bar{x}_2 is treated as a parameter, a constant which it is.

The above equation can be solved for the optimal quantity of factor 1, x_1^* that the firm will use to achieve highest profits. We call x_1^* **the factor demand** for input 1. Just as in the consumer theory, it will be a function of the prices in general, i.e. $x_1^* = x_1^*(p, w_1, w_2)$.

This equation has a very nice economic interpretation. Remember what is $f_1(x_1, \bar{x}_2)$ (the derivative of the production function with respect to x_1) – it is simply the slope of the production function at x_1 , or in other words - **the marginal product of factor 1**.

Thus equation (1) is saying that **at optimum** we must have:

$$pMP_1 = w_1 \tag{2}$$

What does this mean? Remember what was the marginal product (MP_1) - it was the increment in output obtained by hiring one more unit of input 1. Multiply this by the price of output and the left hand side above can be interpreted as the **value of marginal product** of input one, i.e. the increment in revenue (money) that we would obtain if hiring one more unit of input 1. What is the right hand side then? It is simply the **cost of hiring an additional unit** of the factor. Thus optimality requires that we set the additional revenue we get from hiring one more unit of the input equal to the cost of obtaining that unit.

Why is this optimal? Well, suppose that $pMP_1 > w_1$ at our profit maximizing optimum, i.e. that the additional revenue obtained from hiring one more unit of the factor is larger than the cost of doing so. Then we can increase our profits by hiring more of the factor until pMP_1 decreases to a level equal to w_1 (remember hiring more units of the input holding the other input constant decreases its marginal product (MP_1) due to the law of diminishing marginal product). Since we are able to increase profits we must not have been profit maximizing. Similarly, if we suppose that $pMP_1 < w_1$ at the profit maximizing point we see that if we reduce our usage of input 1 we can actually increase our profits since this last unit brings us less value (pMP_1) than what we have to pay for it (w_1).

Let us compute the optimal choice of x_1 (the factor demand) for the Cobb-Douglas production function $f(x_1, x_2) = x_1^\alpha x_2^\beta$. The firm's problem is:

$$\max_{x_1} px_1^\alpha \bar{x}_2^\beta - w_1 x_1 - w_2 \bar{x}_2$$

Setting the first derivative with respect to x_1 we get:

$$p\alpha x_1^{\alpha-1} \bar{x}_2^\beta - w_1 = 0$$

since \bar{x}_2 is just a constant. We can solve the above equation for the factor demand, $x_1^*(p, w_1, w_2)$. We have:

$$x_1^{\alpha-1} = \frac{w_1}{p\alpha \bar{x}_2^\beta}$$

or (raising both sides to power $\frac{1}{\alpha-1}$)

$$x_1^* = \left(\frac{w_1}{p\alpha \bar{x}_2^\beta} \right)^{\frac{1}{\alpha-1}}$$

Notice that for the Cobb-Douglas function the factor demand for input 1 depends on w_1 and p but not on the price of the second input, w_2 .

Numerical Example (different from class)

Let us now consider a particular example with a specific production function and prices. Assume that $f(x_1, x_2) = x_1^{1/2} x_2^{1/2}$, $w_1 = 2$, $w_2 = 1$, $p = 4$ and $\bar{x}_2 = 1$.

Proceeding as above, the firm's problem is (just substituting the numbers):

$$\max_{x_1} 4x_1^{1/2}(1)^{1/2} - 2x_1 - (1)(1)$$

Taking the derivative and setting to zero:

$$4\frac{1}{2}x_1^{-1/2} - 2 = 0$$

or,

$$2x_1^{-1/2} = 2$$

or

$$x_1^{-1/2} = 1$$

i.e.

$$x_1^* = 1$$

(you could have obtained the same expression by substituting the specific numerical values used here for $p, \bar{x}_2, w_1, \alpha, \beta$ in the general expression from before).

Having the optimal quantity demanded of the input we can compute the optimal amount of output that will be produced by simply plugging x_1^* into the production function. We get, $y^* = f(x_1^*, \bar{x}_2) = (1)^{1/2}(1)^{1/2} = 1$. We can also find the maximized profit, $\pi^* = py^* - w_1x_1^* - w_2\bar{x}_2 = 4(1) - 2(1) - 1 = 1$. (everything turns out to be equal to one in this example :-)

Profit Maximization in the Long Run

Now consider the long run - i.e. when all factors are variable and hence can be chosen by the firm when deciding how to maximize profits. The difference from before in our example is that both x_1 and x_2 can now be chosen. Thus the firm's profit maximization problem in the long run looks like:

$$\max_{x_1, x_2} pf(x_1, x_2) - w_1x_1 - w_2x_2$$

Notice that there is no \bar{x}_2 anymore as both inputs are to be chosen.

How to solve it?

Now we have a function of two variables and (unlike the consumer's problem) there is no constraint to use to express x_2 in terms of x_1 or vice versa. So what do we do? Well, same as before: take derivatives and set to zero. The difference is that now we have two variables so we have to take the derivative of the function with respect to each of them **holding the other constant**. This is simply taking the **partial derivatives** with respect to each variable and setting them to zero.

The partial derivative with respect to x_1 (holding x_2 constant) is obtained in the same way as before:

$$pf_1(x_1, x_2) - w_1 = 0$$

where $f_1(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_1}$ is the partial derivative of f with respect to x_1 . Similarly, for x_2 we get

$$pf_2(x_1, x_2) - w_2 = 0$$

where $f_2(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_2}$ is the partial derivative of f with respect to x_2 .

Once again the partial derivatives are simply the marginal products of the two factors, i.e. we have at optimum:

$$\begin{aligned} pMP_1 &= w_1 \\ pMP_2 &= w_2 \end{aligned}$$

The interpretation is the same as before - the value of marginal product of each factor must equal its price at optimum otherwise factor usage can be either decreased or increased yielding higher profits.

The above represents a system of two equations that can be solved for the optimal factor demands, x_1^* and x_2^* (notice that MP_1 and MP_2 are functions of x_1 and x_2).

Let us look at our Cobb-Douglas example and derive the factor demands as we did in the SR case. The long run firm's problem is:

$$\max_{x_1, x_2} px_1^\alpha x_2^\beta - w_1 x_1 - w_2 x_2$$

Taking the partial derivatives and setting them to zero we get:

$$\begin{aligned} p\alpha x_1^{\alpha-1} x_2^\beta &= w_1 \\ px_1^\alpha \beta x_2^{\beta-1} &= w_2 \end{aligned} \tag{3}$$

Notice that these are two equations in two unknowns (x_1 and x_2). How to solve them? The easiest way is to divide them through (i.e. divide both sides of the first equation by the respective sides of the second):

$$\frac{p\alpha x_1^{\alpha-1} x_2^\beta}{px_1^\alpha \beta x_2^{\beta-1}} = \frac{w_1}{w_2}$$

Cancelling the p 's and collecting the powers of x_1 and x_2 we get:

$$\frac{\alpha x_2}{\beta x_1} = \frac{w_1}{w_2} \tag{4}$$

(since $x_1^{\alpha-1}/x_1^\alpha = x_1^{-1} = 1/x_1$ and $x_2^\beta/x_2^{\beta-1} = x_2$).

From above we can express x_2 in terms of x_1 :

$$x_2 = \frac{\beta w_1}{\alpha w_2} x_1 \tag{5}$$

Now use this expression and plug in the first derivative equation ($p\alpha x_1^{\alpha-1} x_2^\beta = w_1$) to get:

$$p\alpha x_1^{\alpha-1} \left(\frac{\beta w_1}{\alpha w_2}\right)^\beta x_1^\beta = w_1$$

Notice that we can

or,

$$p\alpha \left(\frac{\beta w_1}{\alpha w_2}\right)^\beta x_1^{\alpha+\beta-1} = w_1$$

Then we have:

$$x_1^{\alpha+\beta-1} = \frac{w_1}{p\alpha\left(\frac{\beta w_1}{\alpha w_2}\right)^\beta}$$

or, raising to power $\frac{1}{\alpha+\beta-1}$:

$$x_1^* = \left(\frac{w_1}{p\alpha\left(\frac{\beta w_1}{\alpha w_2}\right)^\beta}\right)^{\frac{1}{\alpha+\beta-1}} = \left(\frac{p\alpha}{w_1}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{p\beta}{w_2}\right)^{\frac{\beta}{1-\alpha-\beta}}$$

i.e. some messy function of w_1, w_2 and p . We can then use (4) to get x_2^* as an even messier function of the prices.

Alternative Way to Solve (less messy):

Let's now look at the derivatives once again. Remember that $y = x_1^\alpha x_2^\beta$, i.e. we can write the system (3) as:

$$\begin{aligned} p\alpha \frac{y}{x_1} &= w_1 \\ p\beta \frac{y}{x_2} &= w_2 \end{aligned}$$

Thus we have that at optimum:

$$x_1 = \frac{p\alpha y}{w_1} \text{ and } x_2 = \frac{p\beta y}{w_2}$$

Remembering that $y = x_1^\alpha x_2^\beta$ we get an equation for the optimal output y^* :

$$y^* = \left(\frac{p\alpha y^*}{w_1}\right)^\alpha \left(\frac{p\beta y^*}{w_2}\right)^\beta$$

or,

$$(y^*)^{1-\alpha-\beta} = \left(\frac{p\alpha}{w_1}\right)^\alpha \left(\frac{p\beta}{w_2}\right)^\beta$$

i.e. $y^* = \left(\frac{p\alpha}{w_1}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{p\beta}{w_2}\right)^{\frac{\beta}{1-\alpha-\beta}}$ and hence: $x_1^* = \left(\frac{p\alpha}{w_1}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{p\beta}{w_2}\right)^{\frac{\beta}{1-\alpha-\beta}}$, $x_2^* = \left(\frac{p\alpha}{w_1}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{p\beta}{w_2}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}$. The advantage in this method is that everything is more symmetrical and it stresses the interdependence between the inputs and the output.

The above is general but pretty incomprehensible (although you can simplify it a bit). Let's look at a:

Numerical Example:

Assume that $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$, $w_1 = 1$, $w_2 = 2$, $p = 3$. Then the firm's problem is:

$$\max_{x_1, x_2} 3x_1^{1/3} x_2^{1/3} - x_1 - 2x_2$$

Taking the partials and setting them to zero we get:

$$\begin{aligned}3\frac{1}{3}x_1^{-2/3}x_2^{1/3} &= 1 \\3x_1^{1/3}\left(\frac{1}{3}x_2^{-2/3}\right) &= 2\end{aligned}$$

As in the general case we have two equations in the two unknowns, x_1 and x_2 . The simplest way to proceed is once again to divide the equations through:

$$\frac{x_2}{x_1} = \frac{1}{2}$$

i.e. $x_2 = x_1/2$. Substitute that into the first equation to get:

$$x_1^{-2/3}(x_1/2)^{1/3} = 1$$

or

$$(2)^{-1/3}x_1^{-1/3} = 1$$

raising both sides to power 3:

$$x_1^{-1} = 2$$

or, $x_1^* = 1/2$. Thus $x_2^* = x_1^*/2 = 1/4$. Output is: $y^* = (1/2)^{1/3}(1/4)^{1/3} = (1/8)^{1/3} = 1/2$, optimal profit is: $\pi^* = 3y^* - x_1^* - 2x_2^* = 1/2$.