Gravitational Turbulence and Instability of Global AdS

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1512.04383, 1507.02684, 1501,00998, 1410.1880
work in progress, and some review

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Motivations

- Gravity in AdS:
  Global AdS is a "sealed finite box", and gravity is attractive.

  A small initial perturbation with "spread-out" energy density might eventually get concentrated, making the geometry deviate significantly from the vacuum, and eventually form a black hole.

- Boundary CFT:
  Black hole = thermal state. So the process of forming a black hole in bulk AdS seems to be a natural process of thermalization in the boundary theory.

The AdS instability problem

Maybe thermalization happens not *eventually*, but at the shortest possible time scale allowed by dynamics.

For a small perturbation of amplitude ϵ, the strength of gravitational self-interaction $\propto \epsilon^2$, so the shortest time scale to expect a dramatic change is $T \sim \epsilon^{-2}$.

Bizon & Rostworowski (2011):
A full GR numerical simulation shows that a small black hole forms at this time scale (and is expected to swallow the rest of matters in a shorter or comparable time scale).

Generic? The analytical side of the story?

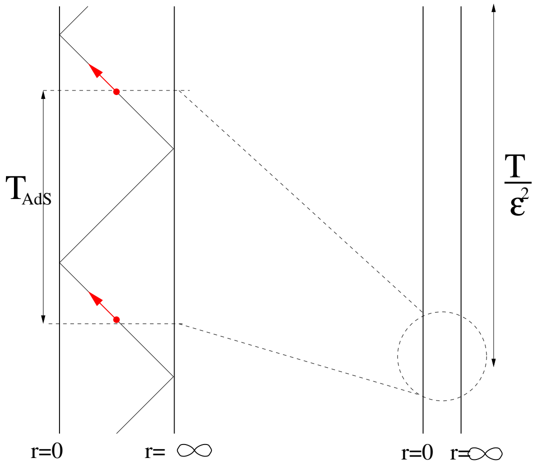
Confusions

- Later simulations showed that there are also initial conditions which do not lead to black hole formation at the $T \sim \epsilon^{-2}$ time scale.
- There is no clear distinction between the two types of initial conditions that lead to these two dramatically different outcomes.
- Some initial conditions that do not form black holes can be understood as being close to special, exactly stable analytical solutions. (1412.4761, 1507.08261 by Buchel, Green, Mailard, Lehner, Liebling)
- It was later realized that black-hole-forming solutions are equally special. (1512.04383 Freivogel and ISY)

Outline

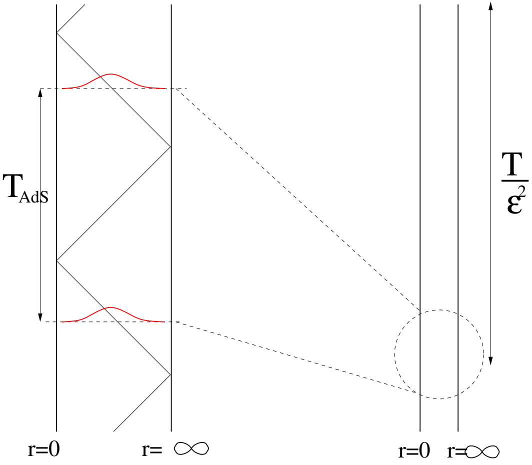
- Review the analytical tools that helped us understood the special solutions:
  Time-averaging + mode-expansion leads to a strongly resonant system with hidden symmetries.
- Understand in what sense are these solutions special, and can we try to guess whether there are more generic solutions?

Time Average



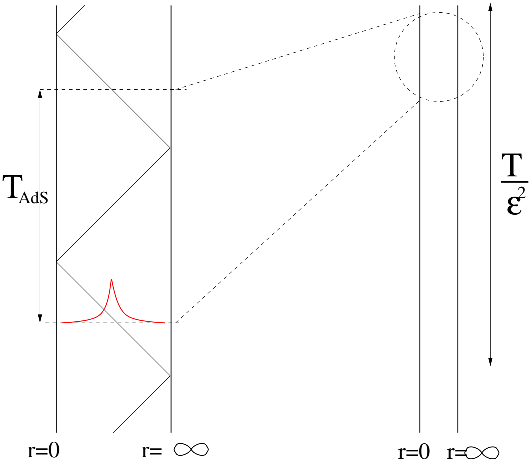
A free theory in AdS background is exactly periodic. Gravitational back-reaction only gives a small correction.

Time Average



Initially, a dilute distribution of energy, may eventually......

Time Average



......becomes a concentrated distribution of energy and forms a black hole.

Time Average

In position space	In Fourier space
Exactly periodic motion	Every eigenmode has integer frequency, $w_n = 2n + d$.
Gravitational interaction	4-mode coupling, $\ddot{\phi}_n + w_n^2 \phi_n = C_{klmn} \phi_k \phi_l \phi_n$
dynamics in the long time	$\phi_n = a_n e^{i w_n t} + a_n^* e^{-i w_n t}$, with a slowly changing $a_n(t)$

$$
\begin{aligned}
2 i w_n \frac{d a_n}{dt} &= \sum_{klm} C_{klmn} a_k a_l a_m^* e^{i(w_k + w_l - w_m - w_n)t} \\
&\approx \sum_{klm}^{k+l=m+n} C_{klmn} a_k a_l a_m^* \ .
\end{aligned}
$$

Rescaling Symmetry

$$2iw_n \frac{da_n}{dt} = \sum_{klm}^{k+l=m+n} C_{klmn} a_k a_l a_m^* \, .$$

A scaling symmetry with $a_n \to (a_n/10)$, $t \to 100t$.

- If an initial condition leads to finite $\Delta g_{\mu\nu}(a_n)$, then initial conditions of the same shape in the zero-amplitude limit guarantees that the back-reaction also approaches zero. This is the necessary and sufficient condition for stability.
- If an initial condition leads to a diverging $\Delta g_{\mu\nu}(a_n)$, then any down-scaled initial condition of the same shape is guaranteed to reach finite back-reaction no matter how small the amplitude is. This is the necessary and sufficient condition for instability.

Conservation Laws

$$2iw_n \frac{da_n}{dt} = \sum_{klm}^{k+l=m+n} C_{klmn} a_k a_l a_m^* \ .$$

- Energy $= \sum_n w_n^2 a_n a_n^*$.
- Particle Number $= \sum_n w_n a_n a_n^*$.
- Coupling energy $= \sum_{klmn} C_{klmn} a_k a_l a_m^* a_n^*$.

Interesting, but a finite number of constraints have not yet led to quantitative insights on the dynamics.

Attractor solutions?

$$2iw_n \frac{da_n}{dt} = \sum_{klm}^{k+l=m+n} C_{klmn} a_k a_l a_m^* \ .$$

Universal behavior $\leftarrow$ Attractor solutions.

Attractor requires stationarity, but

$$2iw_n \frac{da_n}{dt} = 0 \ .$$

is too restrictive.

Attractor solutions

As complex phases and the real amplitudes, $a_n = A_n e^{iB_n}$,

$$2w_n \frac{dA_n}{dt} = \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \sin\left(B_m + B_n - B_k - B_l\right)$$

$$2w_n A_n \frac{dB_n}{dt} = \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \cos\left(B_m + B_n - B_k - B_l\right)$$

$A_n = const.$ with $B_n = f(t) + w_n * g(t)$ is already enough for stationarity.

Attractor solutions

$A_n = const.$ and $B_n = f(t) + w_n * g(t)$ automatically solves

$$2w_n \frac{dA_n}{dt} = \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \sin\left(B_m + B_n - B_k - B_l\right) = 0$$

We just need to find the right combination of A_n such that

$$\frac{dB_n}{dt} = (2w_n A_n)^{-1} \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \propto (\dot{f} + w_n * \dot{g})$$

Quasi-periodic stable solutions

$$\frac{dB_n}{dt} = (2w_n A_n)^{-1} \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \propto (\dot{f} + w_n * \dot{g})$$

With a finite cutoff $n < N_{max}$, after normalization, there is always the same number of equations as the number of unknowns. Discrete sets of solutions exist, and some of them appear to be cutoff independent. (Buchel, Green, Mailard, Lehner, Liebling)

By the nature of cutoff independence, A_n for large n are exponentially suppressed, therefore these solutions guarantees the lack of small scale structures. These are “stable” solutions that will not form black holes.

Power-law collapsing solutions

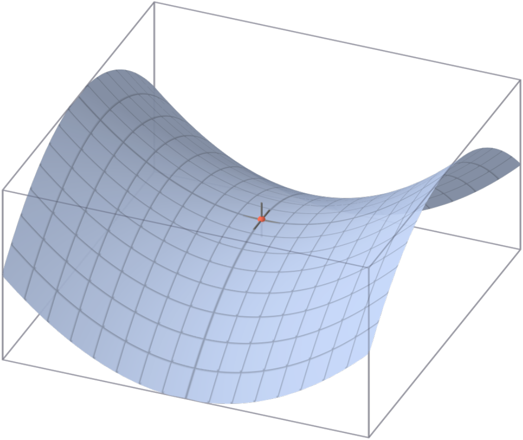
$$\frac{dB_n}{dt} = (2w_n A_n)^{-1} \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \propto (\dot{f} + w_n * \dot{g})$$

Based on the scaling behavior $C_{(\lambda k)(\lambda l)(\lambda m)(\lambda n)} = \lambda^\beta C_{klmn}$, one can derive a power law solution $A_n \propto n^{-\beta/2}$ that solves this asymptotically for large n. (Ben Freivogel and ISY)

In $d > 3$, we have $\beta = d$, and one can show that the phases $B_n \propto f(t) + w_n g(t)$ and amplitudes $A_n \propto n^{-d/2}$ combine to give a diverging back-reaction near $r = 0$.

This is NOT a saddle point

Both stable and unstable solutions have equal presence in numerical results, and equal structures in analytical descriptions. This is very stange!



Coherent phases

Give a system of many nonlinearly interacting harmonic oscillators, $a_n = A_n e^{iB_n}$, we expect typical states to have no phase coherence.

$$\begin{aligned} \langle e^{iB_n} e^{-iB_m} \rangle &= \delta_{mn} \ , \\ \langle e^{iB_k} e^{iB_l} e^{-iB_m} e^{-iB_n} \rangle &= \delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm} \ , \\ &\ldots\ldots \end{aligned}$$

The two solutions we saw earlier, one back-reacts strongly and the other does not, but they both have coherence phases:

$$\langle e^{iB_k} e^{iB_l} e^{-iB_m} e^{-iB_n} \rangle = \delta_{(k+l)(m+n)} \ .$$

Human bias toward phase coherence

Popular initial conditions in numerical simulations:

- Two-mode initial data: $a_0, a_1 \neq 0$.
  Two points form a line, so by-definition this is phase coherent.
- Initial $\phi(x,0) = f(x)$, and $\dot{\phi}(x,0) = 0$.
  This means choosing all initial $B_n = 0$, which is coherent!

Sub-consciously, we favor phase-coherent initial conditions. It is possible that many numerical simulations only explore a very biased and small subspace of initial conditions.

Random phase dynamics

When the phases are random, we can derive the dynamics about the amplitudes N_n defined by $\langle a_m a_n^* \rangle = (N_n/w_n)\delta_{mn}$. This is a standard exercise in weakly turbulent systems.

$$\begin{aligned} \dot{N}_n &= w_n \left(\langle \dot{a}_n a_n^* \rangle + \langle a_n \dot{a}_n^* \rangle \right) \\ &= \frac{-i}{2} \sum_{klm}^{k+l=m+n} C_{klmn} \left(\langle a_k a_l a_m^* a_n^* \rangle - \langle a_k^* a_l^* a_m a_n \rangle \right) = 0 \ . \end{aligned}$$

The first time derivative for the amplitudes is zero due to random phase cancellations. This does not mean the absence of dynamics. This means that all the change happens in the phases, which can potentially deviate from the originally random phases.

Random phase dynamics

Take another derivative, we can study how the dynamics first push the phases away from being completely random, and then how that affects the amplitudes.

$$\ddot{N}_n = - \sum_{klm}^{k+l=m+n} \frac{N_k N_l N_m N_n}{w_k w_l w_m w_n} C_{klmn}^2 \left(N_k^{-1} + N_l^{-1} - N_m^{-1} - N_n^{-1} \right)$$

Stationary solutions:

- Equipartition in energy: $N_n \propto w_n^{-1}$.
- Equipartition in particle number: $N_n = const.$

These are UV-infinite, thus usually excluded.

Random phase dynamics

Take another derivative, we can study how the dynamics first push the phases away from being completely random, and then how that affects the amplitudes.

$$\ddot{N}_n = -\sum_{klm}^{k+l=m+n} \frac{N_k N_l N_m N_n}{w_k w_l w_m w_n} C_{klmn}^2 \left(N_k^{-1} + N_l^{-1} - N_m^{-1} - N_n^{-1} \right)$$

Stationary solutions:

- Kolmogorov-Zakharov power law: $C_{(\lambda k)(\lambda l)(\lambda m)(\lambda n)} = \lambda^{\beta} C_{klmn} \to N_n \propto n^{(1-2\beta)/3}$

Leads to a diverging back-reaction near $r = 0$.

Random phase dynamics

Take another derivative, we can study how the dynamics first push the phases away from being completely random, and then how that affects the amplitudes.

$$\ddot{N}_n = - \sum_{klm}^{k+l=m+n} \frac{N_k N_l N_m N_n}{w_k w_l w_m w_n} C_{klmn}^2 \left(N_k^{-1} + N_l^{-1} - N_m^{-1} - N_n^{-1} \right)$$

Stationary solutions:

- Cut-off independent solutions with exponential tails?

In-principle exists. Work in progress to find these explicit solutions and check their stability.

Conclusion (for now)

- A set of powerful tools to study the long-term dynamics of nonlinear gravitational dynamics in the zero-amplitude limit.
- Unstable (black-hole-forming) and stable (remaining dilute) solutions remain in equal status, which is very curious.
- The random-phase dynamics, a full analogy to weak-turbulence, may provide more insights.