Teaching physics with coupled pendulums

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A careful look at an intricate stage set reveals both the artistic and mechanical skills of the designer. It may seem presumptuous to expect a set designer to include the physics of coupled oscillators in his mechanics background. But an incident during a presentation of the musical *My Fair Lady* at our school suggests that a set designer should be aware of the possibility of coupled oscillations between physically connected suspended backdrops. An examination of this event encouraged us to study several coupled pendulum systems that are educational as well as entertaining.

The scene in Fig. 1 is in the study of Henry Higgins, the central male character in *My Fair Lady*. A suspended panel depicting a bookcase is located behind each male player. The cables of the suspended panels are attached to a common horizontal steel bar. The bar is connected to cables that can be raised to remove the bookshelves from the scene. The suspended panels form a coupled pendulum system as shown in Fig. 2. When the players were scurrying to their acting positions after a set change, an accidental bump set one of the panels oscillating. Energy from the bumped panel was fed via the cables and horizontal bar to the adjacent panel. Beats developed in the oscillations of the panels and persisted for several minutes because of the very strong coupling. The oscillations were distracting and surprising to the director who was unaware of this physical phenomenon.

The coupled panels are similar to two identical simple pendulums connected horizontally by a rigid rod of negligible mass (Fig. 3).[1] The pendulums may oscillate in any direction, but the coupling is strongest when the bobs move in a plane defined by the equilibrium positions of the strings. In one normal mode of this strong coupling situation, the pendulums oscillate exactly in phase. The frequency is that of a simple pendulum of length L, the distance from the bob to the support. In the other normal mode, the pendulums oscillate exactly out of phase. The frequency is that of a simple pendulum of length H, the distance from the bob to the connecting rod.

As a guide to describing the general motions of the coupled pendulums, consider a simple pendulum of length H. Analyzing the force of gravity and the force of the string acting on the bob, and considering small amplitudes, the equation of motion can be written[2]

$$\ddot{x} = -\frac{g}{H}x \tag{1}$$

where x is the horizontal displacement of the bob from the fixed end and

$$\ddot{x} = \frac{d^2x}{dt^2}$$

The forces on a bob of the coupled pendulums are also due to gravity and the string. Analyzing the forces in the same fashion as for a simple pendulum, we arrive at

$$\ddot{x}_1 = -\frac{g}{H}(x_1 - \Delta) \text{ and} \tag{2}$$

$$\ddot{x}_2 = -\frac{g}{H}(x_2 - \Delta) \tag{3}$$

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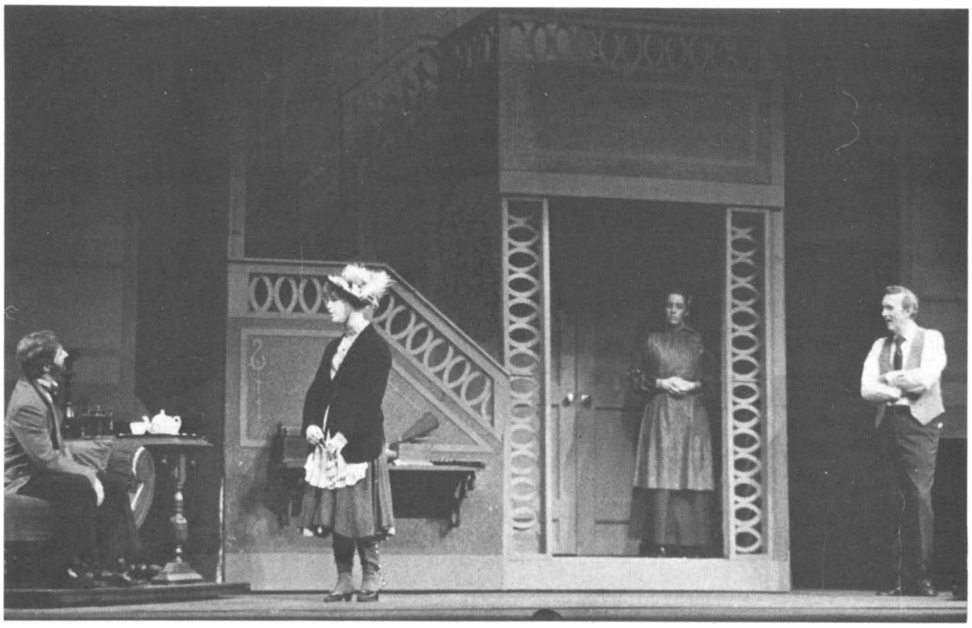

Fig. 1. A scene from the musical *My Fair Lady*. The bookshelves behind the two male players are painted onto flat panels suspended from a common horizontal bar. The suspended panels form a coupled pair of pendulums.

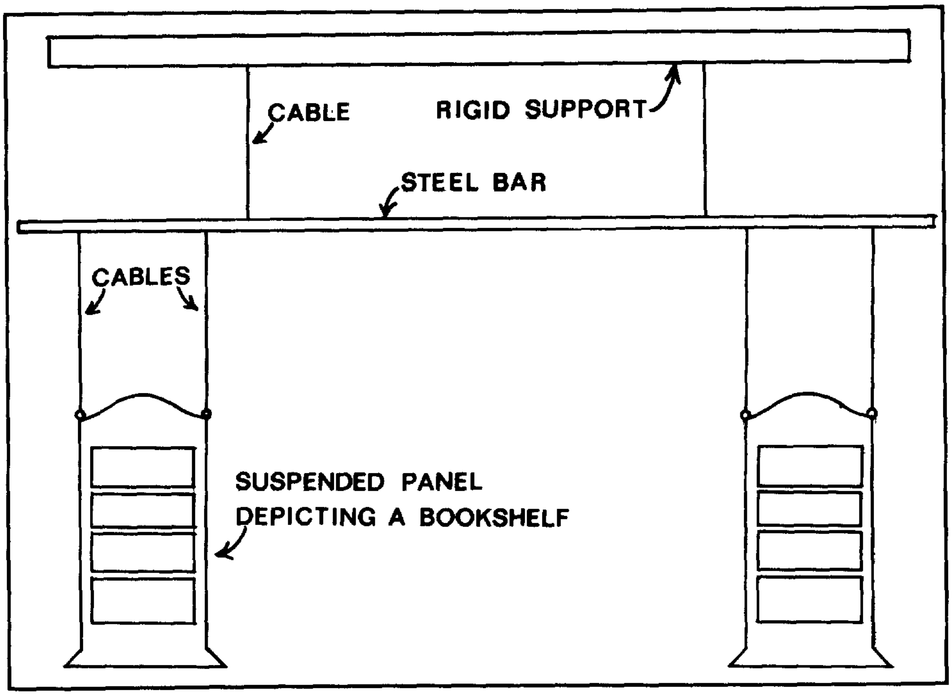

Fig. 2. Details of the construction of the coupled bookshelves shown in Fig. 1.

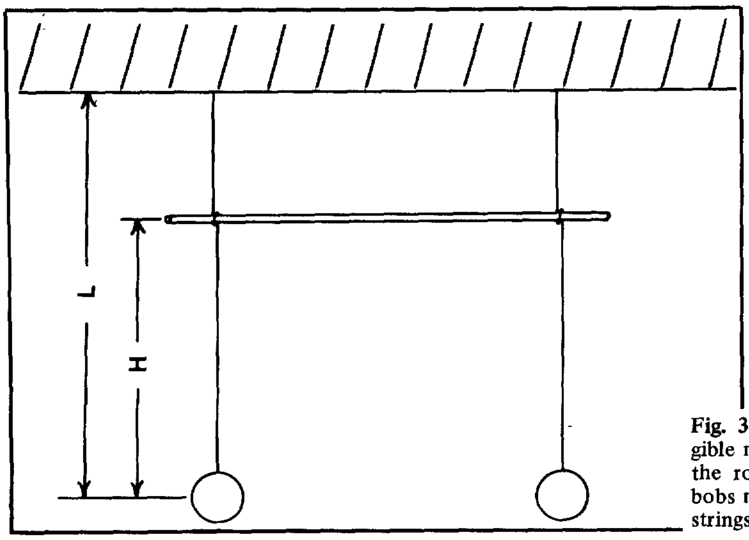

Fig. 3. A coupled pair of pendulums connected by a rod of negligible mass. The strings of the pendulums are simply wrapped around the rod. Coupling between the pendulums is strongest when the bobs move in a plane described by the equilibrium positions of the strings.

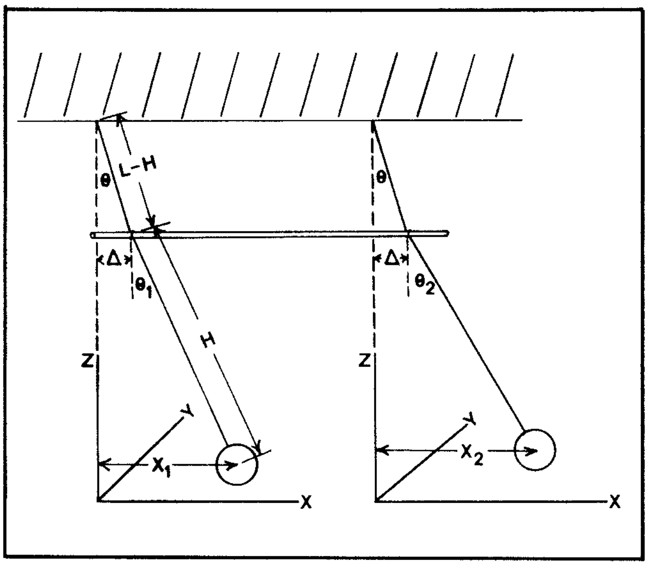

Fig. 4. Designation of the appropriate variables for analyzing the general behavior of the coupled pendulums depicted in Fig. 3. The analysis assumes small oscillations and motion only along the x-axis.

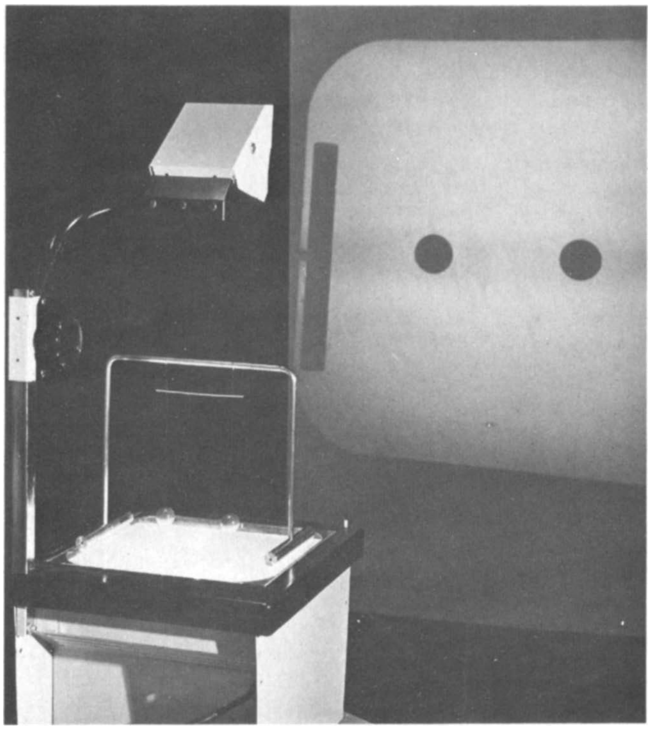

Fig. 5. A working model of the coupled pendulums sketched in Fig. 3. The connecting rod is a wooden medical applicator. Students in the classroom can detect the beats and measure the beat period.

x_1 and x_2 are the horizontal displacements of the bobs from the fixed ends of the pendulums, and Δ is the horizontal distance between a fixed end and the point where the string is attached to the connecting rod (Fig. 4). Δ depends on x_1 and x_2 and reflects the coupling between the pendulums. Solving these coupled differential equations is straightforward. Subtracting Eq. (3) from Eq. (2) yields

$$\ddot{x}_1 - \ddot{x}_2 = -\frac{g}{H}(x_1 - x_2) \quad (4)$$

This equation describes the normal mode for which $x_1 = -x_2$ and the radian frequency is

$$\omega_1 = \sqrt{\frac{g}{H}}$$

Adding Eq. (2) and (3) we have

$$\ddot{x}_1 + \ddot{x}_2 = -\frac{g}{H}(x_1 + x_2 - 2\Delta) \quad (5)$$

To the extent that $\theta_1 \approx \theta_2$ (Fig. 4), then

$$\frac{x_1}{L} \approx \frac{x_2}{L} \approx \frac{\Delta}{L - H} \quad \text{and} \quad \frac{x - \Delta}{H} = \frac{x}{L}$$

and Eq. (5) becomes

$$\ddot{x}_1 + \ddot{x}_2 = -\frac{g}{L}(x_1 + x_2) \quad (6)$$

This equation describes the normal mode for which $x_1 = x_2$ and the radian frequency is

$$\omega_2 = \sqrt{\frac{g}{L}}$$

Setting $x_1 = A_o$, $x_1 = 0$, $x_2 = 0$, $x_2 = 0$ at $t = 0$, the solutions of the coupled equations are

$$x_1 = A_o \cos\left(\frac{\omega_1 - \omega_2}{2}\, t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}\, t\right) \text{and} \quad (7)$$

$$x_2 = A_o \sin\left(\frac{\omega_1 - \omega_2}{2}\, t\right)\sin\left(\frac{\omega_1 + \omega_2}{2}\, t\right) \quad (8)$$

The frequency (ν) and period (T) of the beats are given by

$$\nu = \frac{\omega_1 - \omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{H}} - \frac{1}{2\pi}\sqrt{\frac{g}{L}} \text{ and} \quad (9)$$

$$T = \frac{1}{\nu} \quad (10)$$

It is instructional to verify these results in a classroom environment. We have constructed a model of this coupled-pendulum system that can be used either at a lecture table or on an overhead projector. Figure 5 shows the coupled pair of pendulums on the writing surface of an overhead projector with the motions of the pendulum bobs projected onto a viewing screen. Given two length measurements (H and L in Fig. 3) students can calculate the beat frequency from Eq. (9) and the beat period from Eq. (10). From observations of the motions on the screen they may measure the beat period using a timer in the classroom or their own watches. Agreement between theory and experiment is very good.

An interesting experimental exercise can be formulated around this apparatus. Equation (9) for the beat frequency can be written

$$\nu = \frac{\sqrt{g}}{2\pi} H^{-1/2} - \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad (11)$$

L is constant for a given setup, but H can be changed by merely shifting the position of the connecting rod. Thus, one can measure the beat frequency ν as a function of H. Plotting ν against $H^{-1/2}$ yields a straight line of slope $\sqrt{g}/2\pi$ and intercept $(1/2\pi)\sqrt{g/L}$. From measurements of the slope and intercept one can determine the acceleration due to

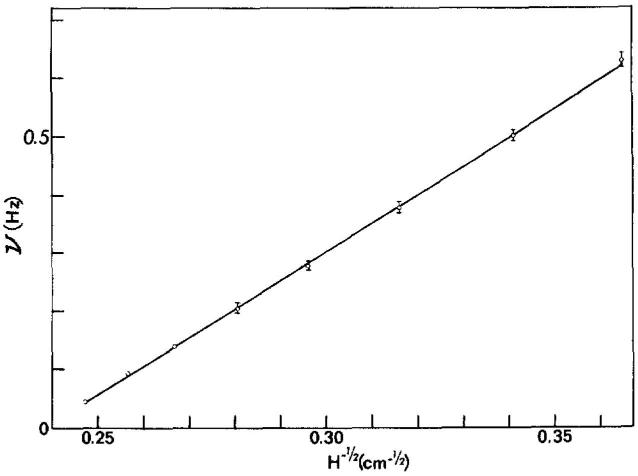

Fig. 6. A plot of the beat frequency (ν) versus $H^{-1/2}$ where H is the distance from the bob to the connecting rod. The error in the beat frequency reflects an estimated uncertainty of 1 mm in the measurement of H. The straight line is a least squares fit to the data. From the slope of the line we deduce $g = 9.6\ \mathrm{m/s^2}$ and from the intercept we find the distance from the bob to the support L to be 17.6 cm. A measurement of L with a meter stick confirms the value deduced from the graph.

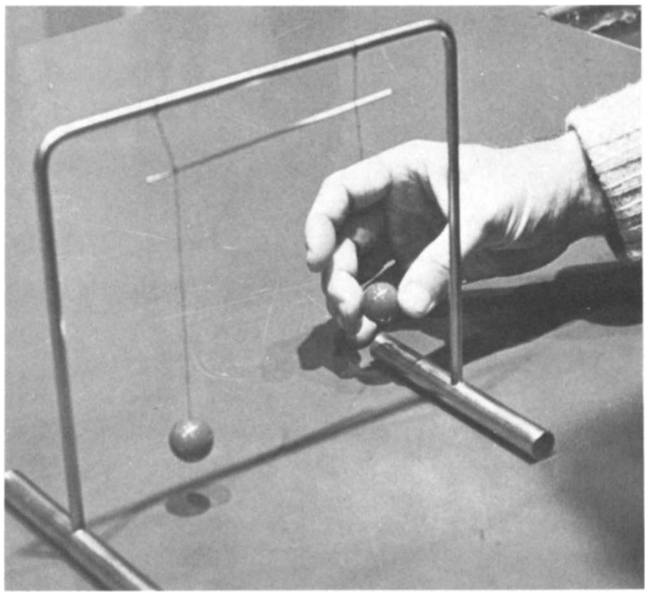

Fig. 7. In this illustration one pendulum is set in motion with both x- and y-components of motion.

gravity and the length L of the pendulums. A set of such measurements and the values deduced for g and L are shown in Fig. 6. The results are quite satisfactory.

There is considerable practical merit to this version of coupled pendulums. For example, many textbooks[3,4] consider a pair of simple pendulums coupled horizontally by a spring between the bobs. While this is an attractive system to treat theoretically, it is cumbersome to work with experimentally. It is awkward to connect a spring between the bobs, and it is very inconvenient to change the spring in order to vary the strength of the coupling. The model described here is extremely easy to construct, and it is trivial to change the coupling because the string is simply wrapped around the connecting rod. It is also easy to increase the number of oscillators.

This coupled pendulum model allows the generation of rather unique Lissajous figures.[3,4] We have established an x-axis as horizontal in the plane containing the equilibrium positions of the string. Let us label the y-axis perpendicular to this plane (Fig. 4). If one pendulum is set in motion in the y-direction it oscillates independently of the other pendulum because the coupling is very weak. Starting with one pendulum at rest and the other oscillating in the y-direction, the motions of the bobs are described by

$$y_1 = A_{oy} \cos \omega_2 t \tag{12}$$

$$y_2 = 0 \tag{13}$$

If now one pendulum is set in motion with both x- and y-components (Fig. 7) then the x-components of the pendulums couple strongly but the y-components are independent. The components of motion of the bobs for this situation are

$$x_1 = A_{ox} \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \tag{14}$$

$$y_1 = A_{oy} \cos \omega_2 t \tag{15}$$

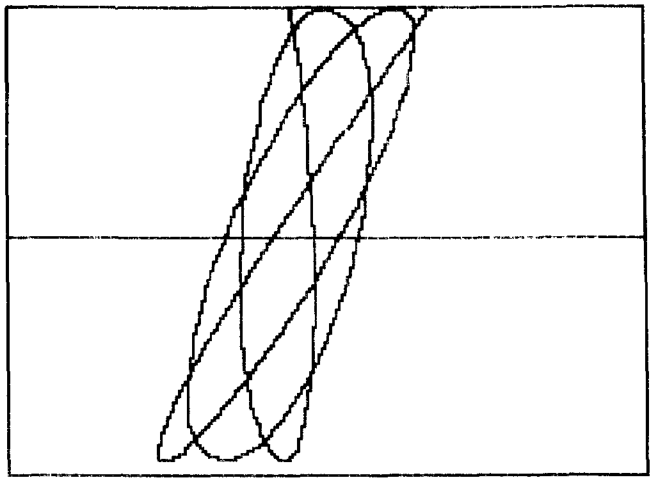

Fig. 8. A computer plot of the trajectory of a bob using Eq. (11) and (12) and values of 13.2 cm and 17.9 cm for H and L.

$$x_2 = A_{ox} \sin\left(\frac{\omega_1 - \omega_2}{2} t\right) \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \tag{16}$$

$$y_2 = 0 \tag{17}$$

The pendulum at rest initially oscillates only along the x-axis, developing beats as it moves. The other pendulum, having a y-component and an x-component whose amplitude changes with time, traces out a Lissajous figure as it oscillates. Figure 8 shows a calculated plot of the figure and Fig. 9 shows a photograph of the pattern. While the formation of this Lissajous figure involves two perpendicular simple harmonic motions, one of the components involves

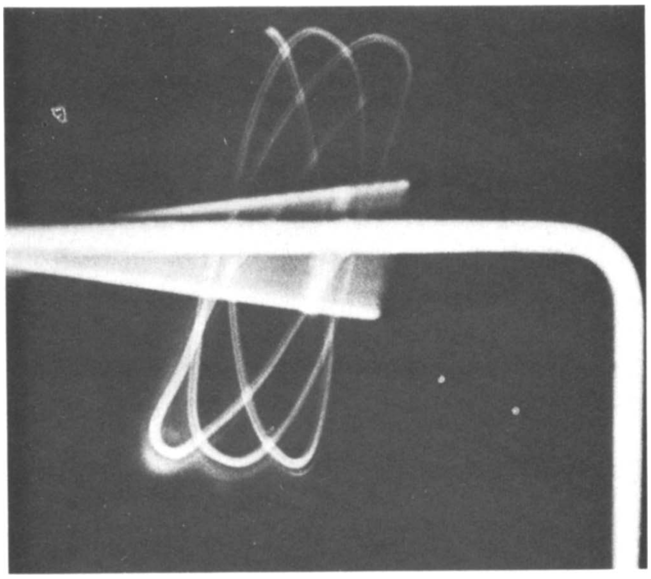

Fig. 9. This photograph records the trajectories of the bobs when one bob is set in motion as shown in Fig. 7. Light for the exposure came from a dot of luminous paint on the top of each bob.

Fig. 10. A coupled pair of pendulums mounted on the wall in a hallway attracts the attention of passersby. Some experiment, others offer explanations, and a few make measurements.

a time-varying amplitude, a feature not present in the production of most Lissajous patterns.

To share the fascination of coupled pendulums and promote interest in physics, we have mounted another model of the coupled pendulums on the wall separating our offices (Fig. 10). Students passing through the hallway or waiting to see us will experiment with the pendulums, sometimes offering explanations and occasionally taking measurements.

The classroom demonstration depicted in Fig. 11 involves students riding in two coupled swings. This version is similar to the coupled backdrops in the *My Fair Lady* stage set. As is the case with the smaller models, coupling between the swings is strongest when the passengers oscillate sideways. The students feel the formation of beats when one swing is moved laterally. When one swing is shoved in an oblique fashion, the passenger traces a Lissajous figure because of the strong horizontal coupling. Beats are observed in the other swing that moves essentially in a horizontal direction. Experiencing these motions and the

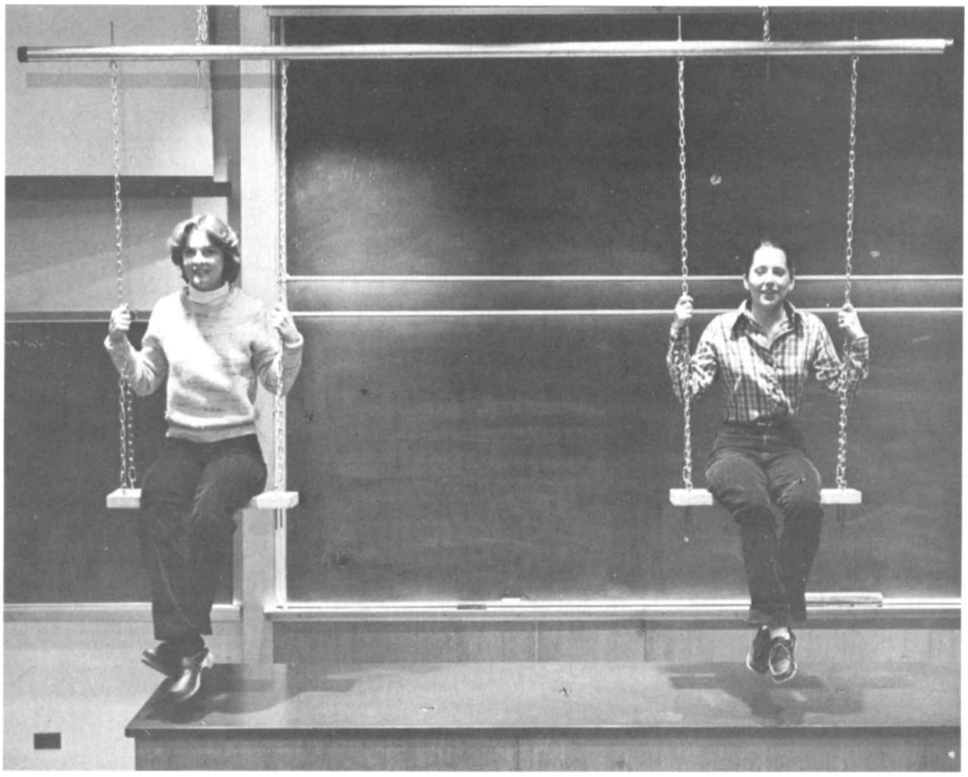

Fig. 11. Two swings connected to a horizontal steel pipe attached by two chains to the ceiling of a classroom form a coupled pair of pendulums for a classroom demonstration. The passengers sense the formation of beats and the interesting motions resulting from setting one swing moving at an oblique angle.

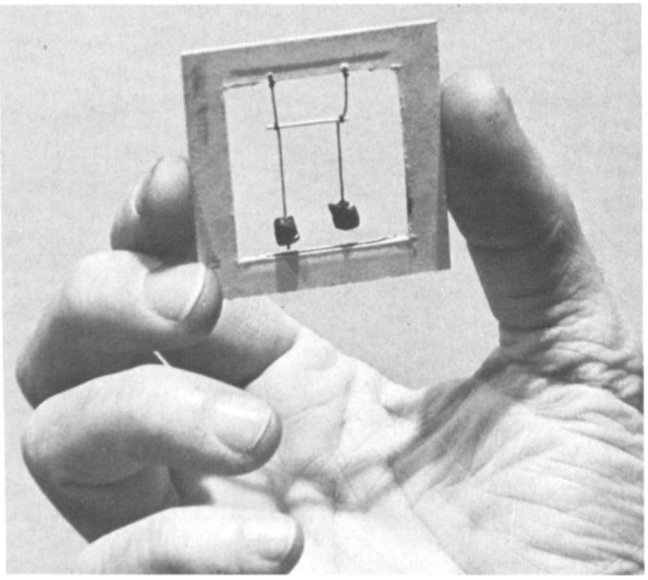

Fig. 12. A small coupled pair of pendulums suspended inside the frame of a 35-mm slide. One pendulum is pulled aside and stuck to the side of the frame with a low melting point material. When dropped into the projector, heat from the lamp melts the material and frees the pendulum bob.

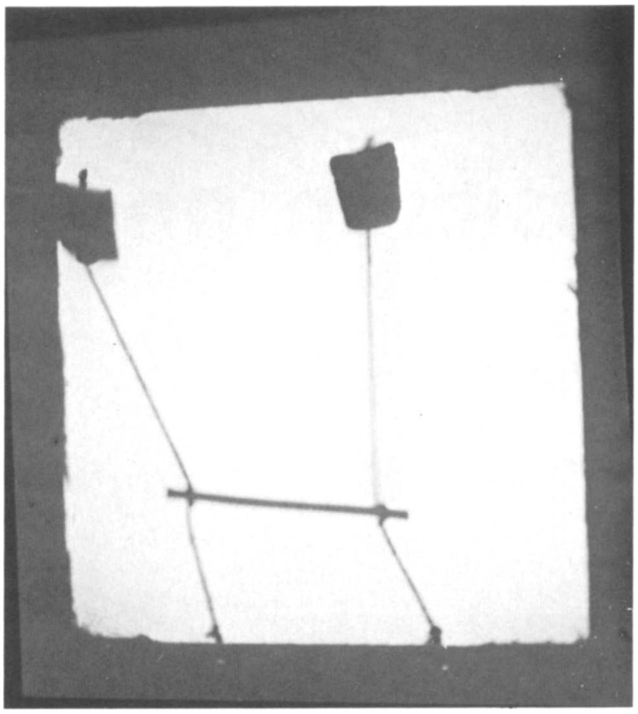

Fig. 13. Projection of the 35-mm slide shown in Fig. 12. The vertical orientation of the coupled pendulums is inverted by the optics of the projector.

formation of beats is an exhilarating experience. It is fun to ride the coupled swings, and there is no problem getting students to participate in a classroom demonstration.

Tiny pendulums with lead bobs suspended inside the opening of a 35-mm slide frame constitute a novel coupled pair of oscillators (Fig. 12). One pendulum is pulled aside and stuck to the side of the frame with a low-melting point material such as wax. When the slide is dropped into the projector, heat from the projection lamp melts the material, releasing the pendulum bob. Beats in the motion of the coupled pendulums are observed on a screen (Fig. 13). Although the projector optics invert the vertical orientation of the pendulums, the novelty of the scheme and the interest it attracts warrants some classroom time.

What better toy than one demonstrating some physics while entertaining the user! The variety of tantalizing motions of a pair of coupled pendulums is fascinating, and the physics is interesting and challenging. It is with these thoughts that we offer the model in Fig. 14. One could call this version an executive toy, for it is a conversation piece when placed on a desk or in a conspicuous spot at home.

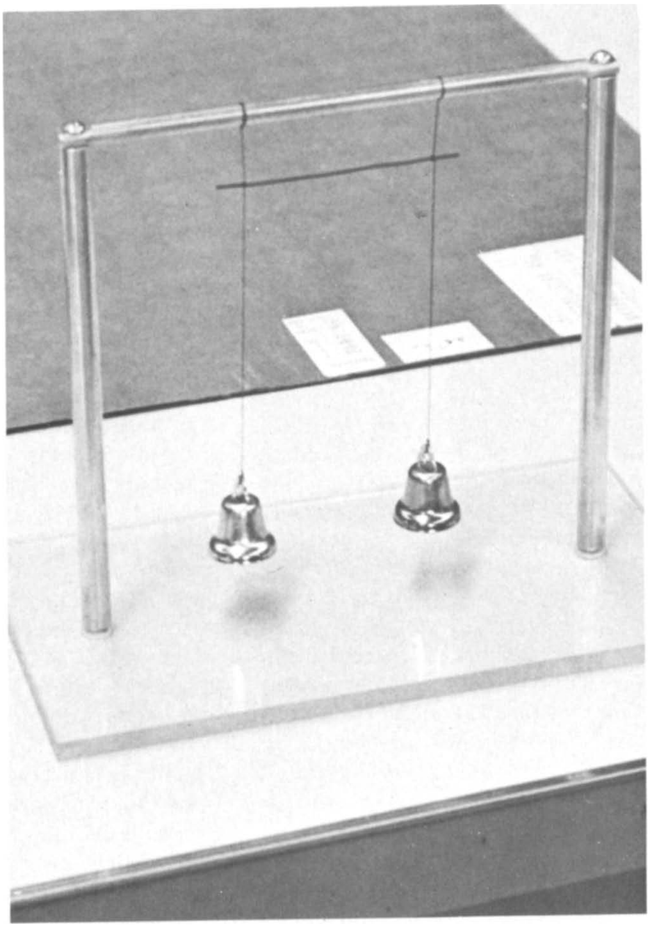

Fig. 14. A coupled pendulum toy. Whether or not one understands the physics, it is fun to watch the variety of tantalizing motions of the pendulum bobs.

References

1. This arrangement of coupled pendulums is described in *Physics Demonstration Experiments*, Vol. 1, Harry F. Meiners (editor) (Ronald Press, New York, 1970). Their behavior is demonstrated in the 8-mm filmloops entitled "Coupled Oscillators" prepared by Alan Holden, Bell Telephone Laboratories, Inc., for the College Physics Film Program of Educational Sciences, Inc. (November 1964). The loops are available from ESB Edcom Division, 19 W. College Avenue, Yardley, PA 19067.
2. See, for example, *Fundamental Physics*, 2nd Ed., David Halliday and Robert Resnick (Wiley, New York, 1981), p. 228.
3. *Vibrations and Waves*, A. P. French (Norton, New York, 1971).
4. *The Physics of Vibrations and Waves*, H. J. Pain (Wiley, New York, 1968).