Simon Fraser University

Economics 798 - Fall 2015

Final Exam

This is a closed book exam. You have TWO HOURS to complete all questions. ALWAYS EXPLAIN your answers.

1. For *any* two convex sets A and B in $\mathbb{R}^2$ with $A \cap B \neq \varnothing$ explain whether the following is true or false:

(a) the set $A \cup B$ is convex

(b) the set $A \setminus B$ is convex

(c) the set $A \cap B$ is convex

2. Consider the matrix $M = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. Do its columns form a basis in $\mathbb{R}^2$? How about in $\mathbb{R}^3$? Find **all solutions** to the linear system $Mx = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $x \in \mathbb{R}^3$.

3. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$ from $\mathbb{R}^2$ to $\mathbb{R}$.

(a) compute the gradient and Hessian of f. Is the Hessian positive or negative (semi-) definite?

(b) Prove that $f(x, 2)$ is continuous and differentiable at $x = 1$. Compute $\frac{\partial f(1,2)}{\partial x}$.

[Reminder: a function $g(x)$ is continuous at point a if $\forall \varepsilon > 0$ there exists a $\delta > 0$ such that $|g(x) - g(a)| < \varepsilon$ for all x for which $|x - a| < \delta$. A function is differentiable at a if $\lim_{t \to 0} \frac{f(a+t) - f(a)}{t}$ exists.]

4. Solve the following constrained maximization problem **using either the Lagrange or the Kuhn-Tucker method**.

$$\max_{a,b} \; a\sqrt{b}$$
$$\text{s.t. } a^2 + b^2 \leq 3$$
$$a \geq 0 \text{ and } b \geq 0$$

5. Suppose the weather can be either {sunny}, {cloudy without rain}, or {rainy} with equal probability. Assign the numbers $1, 0$ and -2 to these outcomes (call them 'weather scores'), respectively and define two random variables: X = the product of the weather scores today and yesterday, and Y = the sum of the weather scores today and yesterday. *Make sure to show all your calculations!*

(a) find the supports of X and Y and write (a table suffices) the joint probability function $f(x, y)$

(b) find the marginal probability functions $f_x(x)$ and $f_y(y)$ and the conditional probability function $f(y|X = 0)$.

(c) compute $Var(Y)$, $E(X)$, and $cov(X,Y)$. Are the random variables X and Y^2 independent?

(d) define the discrete random variable $Z \equiv E(X|Y)$ as taking the values $E(X|Y = y_j)$ with respective probabilities $f_y(y_j)$ for any y_j in the support of Y. Prove, using only the definition of expectation, that

$$E(X) = E(Z)$$

6. Optimal investment

A firm has the production function $f(k_t) = Ak_t$ where k_t is the firm's capital stock at time $t = 0, 1, ...,$ $f(k_t)$ is the amount of output produced and $A \geq 1$. The initial amount of capital $k_0 > 0$ is given. The price of output is 1 at all t. Each period the firm decides how much from its revenue $f(k_t)$ to distribute as dividends, $d_t \geq 0$ and how much, $i_t \geq 0$ to invest. The firm's capital stock evolves over time as:

$$k_{t+1} = (1 - \delta)k_t + i_t$$

where $\delta \in (0, 1)$ is the capital depreciation rate. The firm owners maximize their present value utility from the dividends: $\sum_{t=0}^{\infty} \beta^t u(d_t)$ where $u(d_t) = 2\sqrt{d_t}$ and $0 < \beta < 1/A$.

(a) Write down the owners' maximization problem as the sequence problem (SP) of choosing an infinite sequence $\{k_{t+1}\}_{t=0}^{\infty}$ for firm capital.

(b) Determine what is a proper *state variable* for this problem and write down the functional equation (FE) problem corresponding to (SP) from part (a). What is the state space X? The feasibility correspondence Γ? The return function?

(c) Suppose all required assumptions from lecture are satisfied. Solve problem (FE) for the *value function* v and the *optimal policy* g. Use the 'guess and verify' method by guessing $v(x) = C\sqrt{x}$ where x is the state variable you identified in (b) and C is some constant.